ON THE DEGREE OF FANO THREEFOLDS WITH CANONICAL GORENSTEIN SINGULARITIES

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To the memory of Andrei Nikolaevich Tyurin

ABSTRACT. We consider Fano threefolds V with canonical Gorenstein singularities. A sharp bound $-K_V^3 \leq 72$ of the degree is proved.

1. INTRODUCTION

In this paper we study Fano threefolds with canonical Gorenstein singularities. Such varieties appears naturally in the minimal model theory that explains the following result due to Alexeev:

Theorem 1.1 ([1]). *Let Y be a $\mathbb{Q}$-Fano threefold with $\mathbb{Q}$-factorial terminal singularities and Picard number 1. If the anti-canonical model $\Phi_{|-K_Y|}(Y)$ is three-dimensional, then Y is birationally equivalent to a Fano threefold V with canonical Gorenstein singularities and base point free linear system $|-K_V|$.*

G. Fano [7], [8] studied algebraic threefolds $V \subset \mathbb{P}^n$ whose general linear sections $V \cap \mathbb{P}^{n-2}$ are canonical curves. Under some additional assumptions, these varieties also are Fano threefolds with canonical Gorenstein singularities (see [3]).

By the main result of [2] the degree of Fano threefolds with canonical Gorenstein singularities is bounded. An explicit bound $-K^3 \leq 184$ was obtained by Cheltcov [35]. However it is very far from being sharp:

Conjecture 1.2 (Fano, Iskovskikh). *Let V be a Fano threefold with canonical Gorenstein singularities. Then $-K_V^3 \leq 72$.*

A sharp bound of the degree was known under additional restrictions to singularities:

Theorem 1.3 ([24]). *Let V be a Fano threefold with terminal Gorenstein singularities. Then X can be deformed to a nonsingular Fano threefold. In particular, $-K_V^3 \leq 64$.*

Moreover, Mukai's vector bundle method can be applied in this case to obtain a classification of Fano threefolds with terminal Gorenstein singularities [23]. In the canonical case the situation is more complicated: the assertion of Theorem 1.3 is no longer true that shows the following example.

Example 1.4 (cf. [7], [12, Ch. 4, Remark 4.2])**.** Weighted projective spaces $\mathbb{P}(3, 1, 1, 1)$ and $\mathbb{P}(6, 4, 1, 1)$ are Fano threefolds with canonical Gorenstein singularities. Here $-K^3 = 72$, so they cannot be deformed to smooth ones.

The main result of this paper is the following theorem.

Theorem 1.5. *Let V be a Fano threefold with canonical Gorenstein singularities. Then $-K_V^3 \leq 72$. Moreover, if the equality $-K_V^3 = 72$ holds, then V is isomorphic to a weighted projective space in Example* 1.4.

Thus the present paper completely proves Fano-Iskovskikh Conjecture. The following is an immediate consequence of our results.

Corollary 1.6. *Let $V \subset \mathbb{P}^n$ be a normal projective threefold such that the following condition holds:*

(†) *a general hyperplane section $V \cap \mathbb{P}^{n-1}$ is a K3 surface with at worst Du Val singularities.*

If $\deg V > 72$, then V is a cone.

Indeed, according to [3] (see also [36]) the variety V is Gorenstein and the anti-canonical class $-K_V$ is the class of hyperplane section. If the singularities of V are rational, then they are canonical and by Theorem 1.5 $\deg V = -K_V^3 \leq 72$. If the locus of non-rational singularities is non-empty, then it is zero-dimensional and by the main result of [11] we obtain that V is a cone.

Similarly we have

Corollary 1.7. *Let $U \subset \mathbb{P}^n$ be a normal projective threefold such that the following condition holds:*

(††) *a general hyperplane section $U \cap \mathbb{P}^{n-1}$ is an Enriques surface with at worst Du Val singularities.*

If $\deg U > 36$, then U is a cone.

Indeed, assume that U is not a cone. Then according to [36] (cf. [4]), the variety U is $\mathbb{Q}$-Gorenstein, has only canonical singularities, and its anti-canonical Weil divisor $-K_U$ is $\mathbb{Q}$-linearly equivalent to the class of hyperplane section H. This means that $n(K_U + H) \sim 0$ for some $n \in \mathbb{N}$. The divisor $K_U + H$ defines an n-sheeted covering $\pi \colon V \to U$ which is étale over the smooth locus of U. Then the variety V satisfies the conditions of Theorem 1.5 and therefore, $-K_{V'}^3 \leq 72$. Hence, $\deg U = -K_U^3 = -K_V^3/n \leq 36$.

Note that in contrast with Corollary 1.6, the bound in Corollary 1.7 is not sharp. For example it is easy to show that an involution can not act on varieties from Example 1.4 so that the quotient has only canonical singularities and the action is free in codimension two. Therefore, $\deg U \neq 36$.

We give one more consequence of our theorem.

Corollary 1.8. *Let X be a Fano threefold with canonical (not necessarily Gorenstein) singularities. Assume that*

(i) *the image $\Phi_{|-K_X|}(X)$ of the map given by the anti-canonical linear system is three-dimensional;*

(ii) *a general element $F \in |-K_X|$ is irreducible and has at worst Du Val singularities.*

Then $\dim |-K_X| \le 38$ *and the inequality is sharp.*

The proof of the corollary is similar to that of Corollary 4.6 in [1]: consider a $K_X + |-K_X|$-crepant modification $f: (Y, \mathcal{L}_Y) \to (X, |-K_X|)$ of the pair $(X, |-K_X|)$. Then the linear system $\mathcal{L}_Y \subset |-K_Y|$ is base point free and defines a morphism $g: Y \to \mathbb{P}^N$ whose Stein factorization $Y \to V \to g(Y)$ gives us is a Fano threefold V with canonical Gorenstein singularities. By our theorem $\dim |-K_X| \le \dim |\mathcal{L}_Y| \le \dim |-K_V| \le 38$.

The last corollary is an argument justifying the conjecture that the bound in Theorem 1.5 holds for all Fano threefolds with canonical singularities (H. Takagi).

In conclusion, we note that in the case, when the variety V has only cDV singularities, the bound $-K_V^3 \le 72$ is not sharp. We expect in this case the better bound $-K_V^3 \le 64$. The proof can be obtained by a small modification of our method. On the other hand, the following conjecture looks very realistic:

Conjecture 1.9 (cf. Theorem 1.3). *Let V Fano threefold with only cDV singularities. Then V has a smoothing by a small deformations. In particular, $-K_V^3 \le 64$.*

Remark 1.10. Rationality questions for Fano threefolds satisfying conditions of Theorem 1.5 were discussed in [26].

1.11. We illustrate the proof of Theorem 1.5 on its two-dimensional analogue. More precisely, we reprove a well-known fact: the degree of a del Pezzo surface V with Du Val singularities is bounded by 8 (if V is singular). Let V be a del Pezzo surface with Du Val singularities of degree $K_V^2 = d$. Assume that V is singular. If $d \ge 3$, then the anti-canonical linear system defines an embedding $V \hookrightarrow \mathbb{P}^d$. Consider the linear system $\mathcal{H} \subset |-K_V|$ of hyperplane sections passing through a singular point $P \in V$. Let $\phi: W \to V$ be the minimal resolution. Then $\phi^*(K_V + \mathcal{H}) = K_W + \mathcal{H}_W + B$, where $\mathcal{H}_W$ is the proper transform of $\mathcal{H}$ and B is a nonzero effective divisor. Run the Minimal Model Program with respect to $K_W + \mathcal{H}_W$. Since the linear system $\mathcal{H}_W$ has no fixed components, we do not leave the category of nonsingular surfaces. At the end we get a pair $(X, \mathcal{H}_X)$ having a nonbirational $K_X + \mathcal{H}_X$-negative contraction. There are two possibilities.

a) $X \simeq \mathbb{P}^2$. Then $d - 1 = \dim \mathcal{H} = \dim \mathcal{H}_X \le 5$.

b) $X \simeq \mathbb{F}_n$. Then $\mathcal{H}_X$ is a linear system of sections of the projection $\mathbb{F}_n \to \mathbb{P}^1$.

Note that the birational transform of the linear system $|-K_V|$ is contained in $|-K_X|$. In particular, $|-K_X|$ has no fixed components. In case b) this is possible only if $n \le 2$ and then $d = \dim |-K_V| \le \dim |-K_X| \le 8$.

Our proof in the three-dimensional case follows the one outlined above. However it is much more complicated because of numerous technical details. Let us explain three-dimensional case in details.

Definition 1.12. A projective algebraic variety W is called a *weak Fano* variety, if its anti-canonical divisor is a nef and big $\mathbb{Q}$-Cartier divisor.

Let V be a Fano threefold with canonical Gorenstein singularities. Consider its terminal $\mathbb{Q}$-factorial modification $\phi\colon W \to V$ (see Proposition 3.5). Here $K_W = \phi^* K_V$ and W has only terminal $\mathbb{Q}$-factorial Gorenstein singularities. According to [16, Lemma 5.1] any Weil divisor on W is Cartier, i.e., W is factorial. Conversely, for any weak Fano threefold W with terminal factorial singularities, its multiple anti-canonical image $V = \Phi_{|-nK_W|}(W)$, for some $n \in \mathbb{N}$, is a Fano threefold with canonical Gorenstein singularities. Moreover, $K_W = \Phi^* K_V$ and $-K_V^3 = -K_W^3$. Thus the inequality in Theorem 1.5 is equivalent to the same inequality for the degree of weak Fano threefolds W with terminal factorial singularities. On such varieties K-negative extremal rays are completely classified [5]. Simple analysis of the structure of extremal rays shows that, for W, having the maximal degree ≥ 72, the corresponding variety V either is singular along a line or contains a plane. Further, as in our two-dimensional analog the bound of the degree of V is obtained by detailed analysis of the linear system of hyperplane sections passing through this line or plane.

2. Notation, conventions and preliminary results

2.1. In this paper we work over $\mathbb{C}$, the complex number field. All Fano varieties are usually supposed to be three-dimensional. However Lemmas 4.2 and 4.3 hold in arbitrary dimension modulo the Minimal Model Program (MMP). Always when we say that a variety has, say, canonical singularities, it means that singularities are not worse than that. Usually we do not distinguish between Cartier divisors and corresponding invertible sheaves.

Notation. By $\mathbb{F}_n$ we denote a rational scroll (Hirzebruch surface) and by $\Sigma = \Sigma_n$ and $l = l_n$ we denote its minimal section and fiber, respectively. For $n \geq 1$, there exists a contraction $\mathbb{F}_n \to \mathbb{W}_n$ of the minimal section Σ, where $\mathbb{W}_n$ is the cone over a rational normal curve of degree n in $\mathbb{P}^n$. The vertex of this cone will be denoted by $O = O_n$.

2.2. Everywhere below we assume that V is a Fano threefold with canonical Gorenstein singularities. Using the Riemann-Roch formula and Kawamata-Viehweg Vanishing Theorem, it is easy to find the dimension of the anti-canonical linear system (see [29]):

$$\text{(2.3)} \qquad\qquad \dim |-K_V| = -\frac{1}{2}K_V^3 + 2.$$

Denote $g := -\frac{1}{2}K_V^3 + 1$. This number is called the *genus* of V. Thus,

$$-K_V^3 = 2g - 2 \quad \text{and} \quad \dim |-K_V| = g + 1.$$

Theorem 2.4 ([29]). *A general member $L \in |-K_V|$ has only Du Val singularities.*

The following proposition is an easy consequence of Theorem 2.4 and corresponding facts on K3 surfaces [30].

Proposition 2.5 ([12, Ch. 1, Proposition 6.1], [34]). *In the above notation, assume that $\mathrm{Bs}\,|-K_V| \neq \varnothing$. Then one of the following holds:*

(i) *the scheme $\mathrm{Bs}\,|-K_V| = C$ is a (reduced) nonsingular rational curve and a general divisor $L \in |-K_V|$ is also nonsingular along C; in particular, $C \cap \mathrm{Sing}\,V = \varnothing$;*

(ii) $\mathrm{Bs}\,|-K_V| = \{P\}$ *is a point, a general divisor $L \in |-K_V|$ has at P an ordinary double singularity and $P \in \mathrm{Sing}\,V$ (in this case $-K_V^3 = 2$).*

Proposition 2.6 ([35], cf. [12, Ch. 1, §6], see also [15]). *If $\mathrm{Bs}\,|-K_V| \neq \varnothing$, then $-K_V^3 \leq 46$.*

Similar to the nonsingular case (see [12]) one can easily prove the following.

Proposition 2.7. *In the above notation, let $\Phi = \Phi_{|-K_V|}\colon V \dashrightarrow \overline{V} \subset \mathbb{P}^{g+1}$ be the anti-canonical map, where $\overline{V} = \Phi(V)$. Then $\dim \overline{V} \geq 2$ and $\dim \overline{V} = 2$ if and only if $\mathrm{Bs}\,|-K_V| \neq \varnothing$. If $\mathrm{Bs}\,|-K_V| = \varnothing$, then one of the following holds:*

(i) $\Phi\colon V \to \overline{V}$ *is double cover, in this case $\overline{V} \subset \mathbb{P}^{g+1}$ is a variety of degree $g - 1$;*

(ii) $\Phi\colon V \to \overline{V}$ *is an isomorphism.*

Proposition 2.8 ([35], cf. [12, Ch. 2, §2]). *In case* (i) *of Proposition 2.7 the following inequality holds: $-K_V^3 \leq 40$.*

3. Some facts from the minimal model theory

Basic definitions and facts from the minimal model theory can be found in numerous surveys and textbooks (see, e.g., [18]). Here we recall only these facts which are not included in the most of surveys.
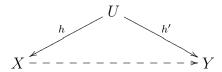
Singularities of linear systems (see [1]). Let X be a normal variety and let $\mathcal{H}$ be a linear system (of Weil divisors) without fixed components on X. For any birational map $\psi\colon X \dashrightarrow Y$, we denote the proper (birational) transform $\psi_*\mathcal{H}$ of $\mathcal{H}$ by $\mathcal{H}_Y$. Sometimes, if it does not cause confusion, we will write $\mathcal{H}$ instead of $\mathcal{H}_Y$. Assume that $K_X + H$ is a $\mathbb{Q}$-Cartier divisor for $H \in \mathcal{H}$. For any good resolution $f\colon Y \to X$ of singularities of the pair $(X, \mathcal{H})$ we can write

$$K_Y + \mathcal{H}_Y = f^*(K_X + \mathcal{H}) + \sum_E a(E, \mathcal{H})E,$$

where E runs through all exceptional divisors and $a(E, \mathcal{H}) \in \mathbb{Q}$. Here and below in numerical formulas we will use notation $\mathcal{H}$, $\mathcal{H}_Y$ and so on, instead of general members of these linear systems. We say that a pair $(X, \mathcal{H})$ has *canonical* singularities (or simply a pair $(X, \mathcal{H})$ is *canonical*), if $a(E, \mathcal{H}) \geq 0$ for all E. If a general member of $\mathcal{H}$ is irreducible, then a pair $(X, \mathcal{H})$ is canonical if and only if so is the pair (X, H), where $H \in \mathcal{H}$ is a general divisor [17, Th. 4.8]. The Log Minimal Model Program works in the category of three-dimensional canonical $\mathbb{Q}$-factorial pairs [1].

Lemma 3.1. *Let $(X, \mathcal{L})$ be a canonical pair and let $X \dashrightarrow Y$ be a birational map such that the inverse map does not contract any divisors (birational 1-contraction in Shokurov's terminology). Assume that $K_X + \mathcal{L} \equiv 0$. Then the pair $(Y, \mathcal{L}_Y)$ is also canonical and $K_Y + \mathcal{L}_Y \equiv 0$.*

Proof. Consider a "Hironaka hut"

Let $\{E_i\}$ be the set of all h'-exceptional divisors. Write

$$K_U + \mathcal{L}_U \;=\; h^*(K_X + \mathcal{L}_X) + \sum a_i E_i, \quad a_i \geq 0,$$

$$K_U + \mathcal{L}_U \;=\; h'^*(K_Y + \mathcal{L}_Y) + \sum b_i E_i.$$

Then $\sum b_i E_i \underset{Y}{\equiv} \sum a_i E_i$. This gives us $a_i = b_i$ (see [37, 1.1]). $\square$

Definition 3.2. Let $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ be log pairs. A birational morphism $f\colon (X, \mathcal{L}) \to (X', \mathcal{L}')$ is said to be $K + \mathcal{L}$-*crepant*, if

$$K_X + \mathcal{L} = f^*(K_{X'} + \mathcal{L}'), \qquad \mathcal{L}' = f_*\mathcal{L}.$$

A birational map $(X, \mathcal{L}) \dashrightarrow (X', \mathcal{L}')$ is said to be *crepant*, if there exists a normal variety U and a commutative triangle of $K + \mathcal{L}$-crepant birational morphisms

Lemma 3.3 ([17, Lemma 3.10])**.** *Let $(X, \mathcal{L})$ be a canonical pair and let $X \dashrightarrow Y$ be a crepant birational map. Then the pair $(Y, \mathcal{L}_Y)$ is also canonical.*

Lemma 3.4. *Let $(W, \mathcal{H})$ be a canonical pair. Assume that*

  (i) *X has only $\mathbb{Q}$-factorial terminal singularities;*
 (ii) *the linear system $\mathcal{H}$ is nef;*
(iii) *$\mathcal{H}$ consists of Cartier divisors.*

Then the steps of the $K + \mathcal{H}$-Minimal Model Program preserve properties (i)-(iii).

Proof. Let $\varphi \colon W \to W'$ be an extremal $K_W + \mathcal{H}$-negative contraction and let (as usual) $\mathcal{H}' = \varphi_* \mathcal{H}$. Since the linear system $\mathcal{H}$ is nef, this contraction is also K_W-negative. Therefore the property (i) is kept under divisorial contractions and flips.

Assume that the contraction φ is divisorial. Then the variety W' is terminal. Let C be a contractible curve. If $\mathcal{H} \cdot C = 0$, then by properties of extremal contractions, $\mathcal{H} = \varphi^* \mathcal{H}'$ and $\mathcal{H}'$ is a linear system of nef Cartier divisors on W'. Assume that $\mathcal{H} \cdot C \geq 1$. Then $-K_W \cdot C > 1$. In particular, φ contracts a divisor E to a point $P' \in W'$ (see [21, 2.3]). So the linear system $\mathcal{H}'$ is nef. Indeed, if $R' \subset W'$ is an arbitrary curve and $R \subset W$ is its proper transform, then $\mathcal{H}' \cdot R' \geq \mathcal{H} \cdot R \geq 0$. For a general divisor $H \in \mathcal{H}$, the restriction $\varphi|_H \colon H \to H' = \varphi(H)$ is a birational contraction of normal surfaces with Du Val singularities (because pairs $(W, \mathcal{H})$ and $(W', \mathcal{H}')$ are canonical). By the Adjunction Formula the divisor $-K_H = -(K_W + H)$ is $\varphi|_H$-ample. Hence discrepancies of all exceptional over $P' \in H'$ divisors are strictly positive [18, Lemma 3.38]. This means that the point $P' \in H'$ is nonsingular. Therefore so is the point $P' \in W'$ (see [37, Corollary 3.7]) and H' is a Cartier divisor at P'.

Assume that the contraction φ is small (flipping). Consider the flip $W \xrightarrow{\varphi} W' \xleftarrow{\psi} W^+$. Let C be an exceptional curve. According to [21, 2.3] we have $-K_W \cdot C < 1$. Hence $\mathcal{H} \cdot C = 0$. As above, by properties of extremal contractions, $\mathcal{H} = \varphi^* \mathcal{H}'$ and $\mathcal{H}'$ is a linear system of nef Cartier divisors on W'. Therefore properties (ii)-(iii) hold for $\mathcal{H}^+ = \psi^* \mathcal{H}'$. $\qquad\square$

Terminal modification.

Proposition-definition 3.5 ([28], [21], [18, Th. 6.23, 6.25])**.** *Let X be a threefold with only canonical singularities. Then there exists a threefold Y with only $\mathbb{Q}$-factorial terminal singularities and a birational contraction $f \colon Y \to X$ such that $K_Y = f^* K_X$. Such an f is called a terminal $\mathbb{Q}$-factorial modification of X.*

4. Contractions of extremal rays on weak Fano threefolds

To work with Fano varieties it is very convenient to use the following definition which was introduced in the unpublished preprint of V.V. Shokurov and the author.

Definition 4.1. A 0-*pair* is a pair (X, D), consisting of a projective algebraic variety X and a boundary D on X such that

    (i) (X, D) is Kawamata log terminal;
    (ii) $K_X + D \equiv 0$.

We say that 0-pair (X, D) is *generating*, if X is $\mathbb{Q}$-factorial and components of D generate the group $N^1(X)$ of $\mathbb{R}$-divisors modulo numerical equivalence. The following two lemmas are easy consequences of the log Minimal Model Program.

Lemma 4.2 ([27])**.** *Let (X, D) be a generating 0-pair.*

    (i) *There exists a boundary Δ such that the pair (X, Δ) is Kawamata log terminal, the divisor $-(K_X + \Delta)$ ample and $\mathrm{Supp}(\Delta) = \mathrm{Supp}(D)$.*
    (ii) *The Mori cone $\overline{\mathrm{NE}}(X)$ is polyhedral and generated by contractible extremal rays (i.e., rays R, for which there exists a contraction $\varphi_R \colon X \to X'$ in the sense of Mori [20], a morphism φ_R of normal varieties with connected fibers such that the image of a curve C is a point if and only if $[C] \in R$).*
    (iii) *Θ-Minimal Model Program works with respect to any (not necessarily effective) divisor Θ.*

Lemma 4.3 ([27])**.** *Let (X, Δ) be a $\mathbb{Q}$-factorial Kawamata log terminal pair such that the divisor $-(K_X + \Delta)$ is nef and big. Then there exists a boundary D such that (X, D) is a generating 0-pair.*

Lemma 4.4. *Let (X, D) be a generating 0-pair and let M be an integral nef Weil divisor on X. Then $H^i(X, M) = 0$ for all $i > 0$.*

Proof. By Lemma 4.2 there is a boundary Δ such that the pair (X, Δ) is Kawamata log terminal and the divisor $-(K_X + \Delta)$ is ample. Now the statement follows by the Kawamata-Viehweg vanishing theorem. $\square$

Let $R \subset \overline{\mathrm{NE}}(X)$ be a (not necessarily K-negative) extremal ray on a normal projective variety X. Put

$$\mathrm{Ex}(R) = \bigcup_{[C] \in R} C.$$

We say that a contractible extremal ray R is of type $(\mathbf{n}, \mathbf{m})$ if $\dim \mathrm{Ex}(R) = n$ and $\dim \varphi_R(\mathrm{Ex}(R)) = m$. Further, we distinguish cases $(\mathbf{n}, \mathbf{m})^-$, $(\mathbf{n}, \mathbf{m})^+$ and $(\mathbf{n}, \mathbf{m})^{\mathbf{0}}$ according to the sign of $K_X \cdot R$.

Everywhere below we shall assume that W is a weak Fano threefold with $\rho(W) > 1$ having only terminal factorial singularities. According to Lemma 4.3 all the extremal rays on W are contractible. The anti-canonical divisor

$-K_W$ defines a face of the Mori cone $\overline{\mathrm{NE}}(W)$ and its contraction $\phi\colon W \to V$ to a Fano threefold with canonical Gorenstein singularities.

Consider a birational K-negative contraction $f\colon W \to W'$. According to [21, (2.3.2)] it is divisorial.

Proposition-definition 4.5. *Notation as above. Let S be an exceptional divisor. If $f\colon W \to W'$ is of type $(\mathbf{2,0})^-$, then one of the following holds:*

Case $(\mathbf{2,0})^-_\bullet$. *W' also has only terminal factorial singularities and is a weak Fano threefold with $-K^3_{W'} \geq -K^3_W$;*

Case $(\mathbf{2,0})^-_\circ$. *$S \simeq \mathbb{P}^2$, $\mathcal{O}_S(S) \simeq \mathcal{O}_{\mathbb{P}^2}(2)$ and $f(S) \in W'$ is a point of type $\frac{1}{2}(1,1,1)$.*

If $f\colon W \to W'$ is of type $(\mathbf{2,1})^-$, then W' also has only terminal factorial singularities and one of the following holds:

Case $(\mathbf{2,1})^-_\bullet$. *W' is also a weak Fano threefold with $-K^3_{W'} \geq -K^3_W$;*

Case $(\mathbf{2,1})^-_\circ$. *For $C := f(S)$ we have $K_{W'} \cdot C > 0$ and C is the only curve having negative intersection with $K_{W'}$. In this case, there exists a small contraction of C, $C \simeq \mathbb{P}^1$ and W' smooth along C. There are two subcases:*

$$(\mathbf{2,1})^-_{\circ 0} \qquad S \simeq \mathbb{P}^1 \times \mathbb{P}^1;$$
$$(\mathbf{2,1})^-_{\circ 1} \qquad S \simeq \mathbb{F}_1.$$

Proof. In case $(\mathbf{2,0})^-$, the assertion follows by the classification of extremal rays on threefolds with terminal factorial singularities [5].

Consider case $(\mathbf{2,1})^-$. According to [5] the variety W' smooth along C and f is the blow up of C. We have $K_W = f^* K_{W'} + S$ and

$$K^3_W = K^3_{W'} + 3f^* K_{W'} \cdot S^2 + S^3.$$

If W' is a weak Fano threefold, then

$$0 \leq (-K_W)^2 \cdot S = 2f^* K_{W'} \cdot S^2 + S^3, \quad 0 \leq (-K_W) \cdot (-f^* K_{W'}) \cdot S = f^* K_{W'} \cdot S^2.$$

This gives us $K^3_W \geq K^3_{W'}$

Assume now that W' is not a weak Fano threefold. Then $K_{W'} \cdot \Gamma > 0$ for some irreducible curve Γ. Let $\tilde{\Gamma} \subset W$ be a irreducible curve dominating Γ. Since $f^* K_{W'} \cdot \tilde{\Gamma} > 0$ and $K_W \cdot \tilde{\Gamma} \leq 0$, we have $S \cdot \tilde{\Gamma} < 0$. Therefore, $\Gamma = C$. By Lemma 4.3 there exists a boundary D such that (W, D) is a generating 0-pair. Let $D' := f_* D$. Since $K_W + D \sim 0$, we have $K_{W'} + D' \sim 0$ and (W', D') is also a 0-pair. Further, $D' \cdot C < 0$ and C is the only irreducible curve with this property. By Lemma 4.2 the curve C generates an extremal ray and the corresponding contraction is small. From the Kawamata-Viehweg Vanishing Theorem one immediately obtains that $C \simeq \mathbb{P}^1$ (see [21, Corollary 1.3]).

Hence, $S \simeq \mathbb{F}_n$ for some $n \geq 0$. Further,

$$K^2_W \cdot S = -K_{W'} \cdot C + 2 - 2p_a(C) = 2 - K_{W'} \cdot C.$$

It is clear that the restriction $-K_W|_S$ is a section of the fibration $S \to C$. Let Σ be a minimal section and let l be a fiber. Then we can write

$$(4.6) \qquad\qquad -K_W|_S \sim \Sigma + (n + a)l.$$

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Since this divisor is nef, $a \geq 0$. Thus,

$$(4.7) \qquad 0 \leq -K_{W'} \cdot C = K_W^2 \cdot S - 2 = (-K_W|_S)^2 - 2 = n + 2a - 2.$$

This gives us $n + 2a < 2$, i.e., $a = 0$ and $n \leq 1$. $\qquad \square$

Corollary 4.8. *Notation as above. We have the following table:*

	S	$N_{C/W'}$	$K_{W'} \cdot C$	$K_W \cdot \Sigma$	$-K_{W'}^3$
$(\mathbf{2,1})_{\circ\mathbf{0}}^-$	$\mathbb{P}^1 \times \mathbb{P}^1$	$\mathcal{O}(-2) \oplus \mathcal{O}(-2)$	2	0	$-K_W^3 - 2$
$(\mathbf{2,1})_{\circ\mathbf{1}}^-$	$\mathbb{F}_1$	$\mathcal{O}(-1) \oplus \mathcal{O}(-2)$	1	0	$-K_W^3$

Proof. In our two cases, we have $-K_W|_S \sim \Sigma + nl$, see (4.6). From (4.7) we get $-K_{W'} \cdot C = n - 2$. Hence,

$$\deg N_{C/W'} = -K_{W'} \cdot C - 2 = n - 4.$$

Thus,

$$K_W^3 = K_{W'}^3 + 3f^*K_{W'} \cdot S^2 + S^3 = K_{W'}^3 - 3K_{W'} \cdot C + 2 + K_{W'} \cdot C =$$
$$= K_{W'}^3 - 2K_{W'} \cdot C + 2 = K_{W'}^3 + 2(n-2) + 2 = K_{W'}^3 + 2n - 2.$$

$\qquad \square$

Corollary 4.9. *In case $(\mathbf{2,1})_{\circ\mathbf{0}}^-$, the image $\phi(S)$ of the exceptional divisor on V is a line (a rational curve Γ such that $-K_V \cdot \Gamma = 1$) and the variety V is singular along $\phi(S)$.*

In cases $(\mathbf{2,1})_{\circ\mathbf{1}}^-$ and $(\mathbf{2,0})_{\circ}^-$, $\phi(S)$ is a plane (a rational surface Π such that $K_V^2 \cdot \Pi = 1$).

Now we consider contractions of type $(\mathbf{2,1})^{\mathbf{0}}$.

Lemma 4.10. *Let X be a threefold with terminal factorial singularities and let $f\colon X \to X'$ be an extremal contraction of type $(\mathbf{2,1})^{\mathbf{0}}$. Then the variety X' is 2-factorial (the last means that for any integral divisor G on X', $2G$ is a Cartier divisor).*

Proof. Consider a general hyperplane section $H' \subset X'$ and let $H \subset X$ be a proper transform of H'. Pick a point $P \in X' \cap f(E)$. We may assume that X' is a sufficiently small neighborhood of P. Then $P \in H'$ is a Du Val singularity, and $f|_H\colon H \to H'$ is its minimal resolution. Consider also a general section $F' \subset X'$, passing through $f(E)$. Write $f^*F' = F + rE$, where F is the proper transform and $r \in \mathbb{N}$. Restricting to H we get $f_H^*F'_{H'} = F_H + rE_H$. Here $F'_{H'}$ is a general hyperplane section of the singularity $P \in H'$. Hence $\Gamma := rE_H$ is the fundamental cycle of a (Du Val) singularity $P \in H'$. Write $\Gamma = \sum \gamma_i \Gamma_i$. All the components Γ_j are numerically proportional on X. In particular, all the $\Gamma_j \cdot \Gamma$ are non-zero, i.e., the fundamental cycle is relatively anti-ample. Assume that $P \in H'$ is not a singularity of type A_1. Then, by definition of fundamental cycle, $(\Gamma - \Gamma_j) \cdot \Gamma_j > 0$ for all j. So, $0 > \Gamma \cdot \Gamma_j > -2$. Hence, $\Gamma \cdot \Gamma_j = -1$ for all j. On the other hand, $\sum \gamma_j = -\Gamma^2$ is equal to the multiplicity of a rational singularity $P \in H'$, i.e., $\sum \gamma_j = 2$.

This is possible only if $\Gamma = \Gamma_1 + \Gamma_2$ and $P \in H'$ is a singularity of type A_2. Thus we obtain two cases:

(i) $\Gamma = \Gamma_1$ and $P \in H'$ is a singularity of type A_1;

(ii) $\Gamma = \Gamma_1 + \Gamma_2$, $\Gamma_1 \equiv \Gamma_2$ and $P \in H'$ is a singularity of type A_2.

Let F' be an integral divisor on X', let F be its proper transform on X and let $F \cdot \Gamma_i = a$. Consider, for example, case (ii). Then $(F + arE) \cdot \Gamma_i = a + a\Gamma \cdot \Gamma_i = 0$. By the Cone Theorem $F + arE = f^*F'$ and F' is a Cartier divisor. Similarly, in case (i) we get that $2F'$ is a Cartier divisor. $\square$

Contractions to a curve.

Proposition 4.11. *Assume that on W there exists an extremal ray of type* (**3**,**2**). *Then* $-K_W^3 \leq 54$.

Lemma 4.12. *Assume that $\rho(W) = 2$ and the morphism $\phi\colon W \to V$ contracts a divisor to a curve. Let S be a prime divisor on W. Then*

$$-K_W^3 \leq \min(54,\ 4K_W^2 \cdot S).$$

Proof. Put $\bar{S} := \phi_*S$, $d := -K_W^3$, $\delta := K_W^2 \cdot S$ and assume that $d > 4\delta$. Since the variety V is $\mathbb{Q}$-factorial and $\rho(V) = 1$, we have $d\bar{S} \equiv -\delta K_V$. On the other hand, by Lemma 4.10 the divisor $G := 2\phi_*\bar{S}$ is Cartier. Thus, $-K_V \equiv \frac{d}{2\delta}G$, where $\frac{d}{2\delta} > 2$. Let H be an ample generator of the group $\mathrm{Pic}(V) \simeq \mathbb{Z}$. Then $-K_V = rH$, where $r \geq 3$. A general member $H \in |H|$ is a del Pezzo surface with Du Val singularities (see [34]). By the Adjunction Formula, $-K_H \equiv (r - 1)H|_H$. Hence, $H \simeq \mathbb{P}^2$ or H is a quadric in $\mathbb{P}^3$. It immediately follows that V is isomorphic to $\mathbb{P}^3$ or a quadric in $\mathbb{P}^4$. The first case is impossible because V is singular. In the second case, we have $-K_V^3 = 54$. $\square$

Proof of Proposition 4.11. A general fiber W_η of the corresponding contraction is a smooth del Pezzo surface. Therefore, $K_W^2 \cdot W_\eta = K_{W_\eta}^2 \leq 9$. By Lemma 4.12 we have $-K_W^3 \leq 54$. $\square$

5. The conic bundle case

5.1. In this section, we shall assume that W is a weak Fano threefold with terminal factorial singularities such that there exists an extremal contraction $f\colon W \to Z$ from W to a surface (i.e., of type (**3**,**2**)). The main result of this section is the following proposition.

Proposition 5.2. (i) (*cf.* [22], [25]) *The surface Z is smooth and is a weak del Pezzo surface.*

(ii) (*cf.* [6]) *If $f\colon W \to Z$ is not a $\mathbb{P}^1$-bundle, then $-K_W^3 \leq 54$.*

(iii) *If $f\colon W \to Z$ is a $\mathbb{P}^1$-bundle, then $-K_W^3 \leq 64$ with a unique exception:*

(*) $-K_W^3 = 72$, $Z \simeq \mathbb{P}^2$ *and* $W \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(3))$ (*see Example* 1.4).

Proof. (i) According to [5] the surface Z is smooth and W/Z is a (possibly singular) conic bundle. Let $\mathcal{E} := f_*\mathcal{O}_W(-K_W)$. Then $\mathcal{E}$ is a rank 3 vector bundle. Consider its projectivization $\mathbb{P}(\mathcal{E})$ and let M be the tautological divisor on $\mathbb{P}(\mathcal{E})$. There exists an embedding $W \hookrightarrow \mathbb{P}(\mathcal{E})$ such that $-K_W = M|_W$ and each fiber W_z, $z \in Z$ is a conic in the fiber $\mathbb{P}(\mathcal{E})_z$ of $\mathbb{P}(\mathcal{E})/Z$ (see [20], [5]).

Put $L := f_*K_W^2$. For $m \gg 0$ the divisor $m^2 L = f_*(mK_W)^2$ on Z is moveable and big. We have the standard formula $-4K_Z \equiv f_*K_W^2 + \Delta$ (see, e.g., [22]). Here Δ is the discriminant of f (a reduced divisor on Z). Thus,

$$(5.3) \qquad\qquad -4K_Z \equiv L + \Delta.$$

Assume that there exists an irreducible curve $C \subset Z$ such that $K_Z \cdot C > 0$. It follows from (5.3) that $(4K_Z + \Delta) \cdot C = -L \cdot C \leq 0$. Therefore, $\Delta \cdot C < 0$ and $C^2 < 0$. Consider two cases:

C is a component of Δ. Then

$$(5.4) \quad -2 \leq 2p_a(C) - 2 = (K_Z + C) \cdot C \leq$$
$$\leq (K_Z + \Delta) \cdot C = -(3K_Z + L) \cdot C \leq 0.$$

Since $K_Z \cdot C$ is a positive integer, this gives us a contradiction.

C is not a component of Δ. Then

$$(5.5) \quad -2 \leq 2p_a(C) - 2 = (K_Z + C) \cdot C \leq$$
$$\leq (K_Z + \Delta + C) \cdot C = -(3K_Z + L) \cdot C + C^2 < 0.$$

As above we get a contradiction. Therefore, the divisor $-K_Z$ is nef. From (5.3) we obtain that $-K_Z$ is big. This proves (i).

Let us prove (ii). Our proof is completely similar to that of [6], where the same bound was proved for smooth Fano threefolds with a conic bundle structure.

The following lemma is well known, however the author could not find a suitable reference.

Lemma 5.6. *Let $f \colon X \to Z$ be a (possibly singular) conic bundle over a smooth surface. Assume that the discriminant curve Δ is a tree of rational curves. Then $\Delta = \varnothing$ and f is a $\mathbb{P}^1$-bundle (in particular, X is smooth).*

Proof. According to [33, Theorem 1.13] there exists a standard model, i.e., the following commutative diagram

$$
\begin{array}{ccc}
X & \dashleftarrow & X' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
Z & \xleftarrow{\ \sigma\ } & Z'
\end{array}
$$

where f' is a standard conic bundle, σ is a composition of blowups over the finite set $M := f(\mathrm{Sing}(X))$ and $X' \dashrightarrow X$ is a birational map which is an isomorphism over $X \setminus f^{-1}(M)$. The discriminant curve Δ' of a standard

conic bundle f' is contained in $\sigma^{-1}(\Delta) \cup \sigma^{-1}(M)$. Therefore it is also a tree of rational curves. It follows from the Artin-Mumford exact sequence that $\Delta' = \varnothing$, i.e., f' is a $\mathbb{P}^1$-bundle.

Lemma 5.7 (cf. [32], [13, Lemma 4]). *Let $f\colon X \to Z$ be a $\mathbb{P}^1$-bundle over a non-singular surface Z. Assume that there exists (-1)-curve $C \subset Z$ and let $\delta\colon Z \to Z'$ be the contraction of C.*

- (i) *The relative Mori cone $\overline{\mathrm{NE}}(X/Z')$ is generated by classes of two curves: the fiber of the projection f and the minimal section Σ of the scroll $f^{-1}(C)$.*
- (ii) *Let $f^{-1}(C) \simeq \mathbb{F}_n$. Then $K_X \cdot \Sigma = n - 1$.*
- (iii) *If $n = 0$, then there exists a commutative diagram*

$$
\begin{array}{ccc}
X & \xrightarrow{\ \sigma\ } & X' \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\
Z & \xrightarrow{\ \delta\ } & Z'
\end{array}
$$

  where σ is the contraction of $f^{-1}(C) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ to the second generator and f' is a $\mathbb{P}^1$-bundle.
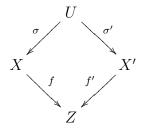- (iv) *If $n = 1$, then there exists a commutative diagram*

$$
\begin{array}{ccccc}
X & \dashxrightarrow{\ \chi\ } & X^+ & \xrightarrow{\ \sigma\ } & X' \\
{\scriptstyle f}\big\downarrow & & & & \big\downarrow{\scriptstyle f'} \\
Z & & \xrightarrow{\ \delta\ } & & Z'
\end{array}
$$

  where χ is a flop, σ is an extremal divisorial contraction of a divisor to a nonsingular point and f' is a $\mathbb{P}^1$-bundle.
- (v) *If $n \geq 1$, then there exists a commutative diagram*

$$
\begin{array}{ccc}
 & U & \\
{\scriptstyle \sigma}\swarrow & & \searrow{\scriptstyle \sigma'} \\
X & & X' \\
{\scriptstyle f}\searrow & & \swarrow{\scriptstyle f'} \\
 & Z &
\end{array}
$$

  where σ is the blowup of Σ, σ' is the contraction of the proper transform of $f^{-1}(C)$ and f' is a $\mathbb{P}^1$-bundle. In this case $\sigma'^{-1}(C) \simeq \mathbb{F}_m$ with $m < n$.

Proof. Put $D := f^{-1}(C)$. The statement of (i) is obvious because $\rho(X/Z') = 2$. (ii) follows from the equalities

$$-2 = (K_D + \Sigma)\cdot\Sigma = (K_X + D)\cdot\Sigma - n = K_X\cdot\Sigma + f^*C\cdot\Sigma - n = K_X\cdot\Sigma + C^2 - n.$$

To prove (iii) one has to check the contractibility criterion of the surface $D \simeq \mathbb{P}^1 \times \mathbb{P}^1$ to the second system of generators. Let us prove (iv) (cf. [14,

§4.1]). Since $K_X \cdot \Sigma = 0$, there exists a flop $\chi \colon X \dashrightarrow X^+$ with center along Σ. Here the anti-canonical divisor $-K_X$ is nef over Z' and the same is true for $-K_{X^+}$. The Mori cone $\overline{\mathrm{NE}}(X^+/Z')$ (as well as $\overline{\mathrm{NE}}(X/Z')$) is generated by two extremal rays. One of them is small and generated by the flopped curve Σ^+. Another one defines a K_{X^+}-negative contraction $\sigma' \colon X^+ \to X'$ over Z'. The variety X^+ is nonsingular [18, Th. 6.15] and the fiber of the projection $X^+ \to Z'$ is two-dimensional and contains D^+, the proper transform of D. It is easy to see that $K_{X^+}^2 \cdot D^+ = K_X^2 \cdot D$ (see [14, §4.1]). Hence,

$$K_{X^+}^2 \cdot D^+ = (K_X + D)^2 \cdot D - 2(K_X + D) \cdot D^2 = K_D^2 - 2K_D \cdot D|_D = 4.$$

According to the classification of extremal rays on nonsingular threefolds [20] the morphism σ' can not contract a divisor to a singular point. Therefore X' is nonsingular. Since f' is smooth outside of the fiber $f'^{-1}(\delta(C))$, f' is smooth everywhere [20, Th. 3.5]. Therefore f' is a $\mathbb{P}^1$-bundle. If $f'(D^+)$ is a curve, then σ' is the blowup of the fiber of the morphism f'. But then $f'^{-1}(\delta(C)) = D^+ \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and on X^+ there is no small contractions. A contradiction proves (iv).

Finally, let us prove (v). Let E be the exceptional divisor of the blowup σ, let D_U be the proper transform of D on U, and let $D' := \sigma'(E)$. It is clear that the divisor D_U satisfies the contractibility criterion onto a curve. Thus there exists a contraction $\sigma' \colon U \to X'$ over Z, where the variety X' is nonsingular. Since the restriction $\sigma'|_E \colon E \to D'$ is an isomorphism, it is sufficient to show that $E \simeq \mathbb{F}_m$ with $m < n$. For the normal sheaf we have a decomposition $N_{\Sigma/X} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$, $a \geq b$. Then $m = a - b$. On the other hand, we have an exact sequence

$$\text{(5.8)} \qquad \begin{array}{ccccccc} 0 & \longrightarrow & N_{\Sigma/D} & \longrightarrow & N_{\Sigma/X} & \longrightarrow & N_{D/X}|_\Sigma & \longrightarrow & 0. \\ & & \| & & & & \| & & \\ & & \mathcal{O}_{\mathbb{P}^1}(-n) & & & & \mathcal{O}_{\mathbb{P}^1}(-1) & & \end{array}$$

This gives us

$$a + b = \deg N_{\Sigma/D} + \deg N_{D/X}|_\Sigma = -n - 1.$$

Hence, $n = -a - b - 1$. If $-n > b$, then in (5.8) the projection $\mathcal{O}_{\mathbb{P}^1}(-n)$ to the second summand of $N_{\Sigma/X}$ is trivial and $-n = a$. Therefore, $b = -1$ and $m = a - b = 1 - n < n$. If $-n \leq b$, then $a + 1 \leq 0$ and

$$m = a - b \leq -1 - b \leq -a - b - 2 = n - 1.$$

$\square$

Now we finish the proof of Lemma 5.6. According to Lemma 5.7 we can contract all the (-1) curves in $\sigma^{-1}(M)$. More precisely, there exists the

following commutative diagram

$$X^0 \dashleftarrow X'$$
$$\downarrow{\scriptstyle f^0} \qquad \downarrow{\scriptstyle f'}$$
$$Z \xleftarrow{\ \sigma\ } Z'$$

where f^0 is a $\mathbb{P}^1$-bundle and birational map $X' \dashrightarrow X^0$ is an isomorphism over $X \setminus f^{0-1}(M)$. Therefore, conic bundles X/Z and X^0/Z are isomorphic in codimension one. But then they are isomorphic everywhere, X is smooth and f is a $\mathbb{P}^1$-bundle. $\qquad\square$

Thus we may assume that in case (ii) the discriminant curve is non-empty.

Lemma 5.9.   (i) $\Delta \sim -3K_Z - c_1(\mathcal{E})$;
(ii) $W \sim 2M + f^*(-K_Z - c_1(\mathcal{E}))$;
(iii) $-K_W^3 = c_1(\mathcal{E}) \cdot (-K_Z + c_1(\mathcal{E})) - 2c_2(\mathcal{E})$.

Proof. The assertion (i) can be found in [33, §1]. For the proof of (ii) we note that $W \sim 2M + f^*G$ for some divisor G on Z. Since $K_{\mathbb{P}(\mathcal{E})} = -3M + f^*(c_1(\mathcal{E}) + K_Z)$, by the Adjunction Formula $G \sim -c_1(\mathcal{E}) - K_Z$. Finally, (iii) follows from (ii) and the fact that $-K_W = M|_W$. $\qquad\square$

We may assume that $-K_V$ is very ample (see [35]). Then the linear system $|-K_W| = |M|\,|_W$ defines a morphism $W \to \mathbb{P}^{g+1}$, which maps fibers f to conics in $\mathbb{P}^{g+1}$. Let $\mathbb{P}_z^2$ be a fiber of $\mathbb{P}(\mathcal{E}) \to Z$ and let $W_z := \mathbb{P}_z^2 \cap W$. Since the linear system $|-K_W|\,|_{W_z}$ defines an embedding, so do $|M|\,|_{\mathbb{P}_z^2}$. Therefore, the linear system $|M|$ has no base points. This means that the vector bundle $\mathcal{E}$ is generated by global sections. In particular, the class $c_1(\mathcal{E}) = -3K_Z - \Delta$ is nef and $c_2(\mathcal{E}) \geq 0$. Thus,

$$-K_W^3 \leq (-3K_Z - \Delta) \cdot (-4K_Z - \Delta) = 12K_Z^2 + 7K_Z \cdot \Delta + \Delta^2.$$

By Lemma 5.6 we have $p_a(\Delta) \geq 1$. Now our assertion is a consequence of the following.

Sublemma 5.10. *Let Z be a smooth weak del Pezzo surface and let Δ be a reduced curve on Z such that $p_a(\Delta) \geq 1$ and the divisor $-3K_Z - \Delta$ is nef. Then $12K_Z^2 + 7K_Z \cdot \Delta + \Delta^2 \leq 54$.*

Proof. Let $\sigma \colon Z \to Z'$ be a contraction of a (-1)-curve L. Put $\Delta' := \sigma(\Delta)$. Then $K_Z = \sigma^* K_{Z'} + L$ and $\Delta = \sigma^* \Delta' - aL$, where $a \geq 0$. Therefore,

$$2p_a(\Delta) - 2 = (K_Z + \Delta) \cdot \Delta = (K_{Z'} + \Delta') \cdot \Delta' + (a-1)aL^2 \leq 2p_a(\Delta') - 2,$$

$$12K_Z^2 + 7K_Z \cdot \Delta + \Delta^2 = 12(\sigma^* K_{Z'} + L)^2 + 7(\sigma^* K_{Z'} + L) \cdot (\sigma^* \Delta' - aL)$$
$$+ (\sigma^* \Delta' - aL)^2 = 12K_{Z'}^2 - 12 + 7K_{Z'} \cdot \Delta' + 7a + (\Delta')^2 - a^2$$
$$\leq 12K_{Z'}^2 + 7K_{Z'} \cdot \Delta' + (\Delta')^2.$$

Thus we may assume that the surface Z contains no (-1)-curves, i.e., Z is isomorphic to $\mathbb{P}^2$ or $\mathbb{F}_n$ with $n = 0$, 2.

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Consider the case $Z \simeq \mathbb{P}^2$. Let $d := \deg \Delta$. Since the divisor $-3K_Z - \Delta$ is nef, we have $3 \le d \le 9$. Hence,

$$12K_Z^2 + 7K_Z \cdot \Delta + \Delta^2 = 108 - 21d + d^2 \le 54.$$

Now consider the case $Z \simeq \mathbb{F}_n$, $n = 0, 2$. Let Σ be the minimal section and let l be a fiber. We can write $\Delta \sim \alpha\Sigma + \beta l$ for some $\alpha, \beta \in \mathbb{Z}$. Since $p_a(\Delta) \ge 1$,

$$0 \le (K_{\mathbb{F}_n} + \Delta) \cdot \Delta = -n\alpha(\alpha - 2) + (\alpha - 2)\beta + \alpha(\beta - 2 - n) =$$
$$(\alpha - 1)(-n\alpha + 2\beta - 2) - 2.$$

In particular, $\alpha \ge 2$. This gives us $-n\alpha + 2\beta - 2 \ge 1$, $2\beta \ge 3 + n\alpha$. Further,

$$12K_Z^2 + 7K_Z \cdot \Delta + \Delta^2 = 96 + 14n\alpha - 14\beta - 7n\alpha - 14\alpha - n\alpha^2 + 2\alpha\beta$$
$$= (7 - \alpha)(n\alpha - 2\beta + 14) - 2.$$

Since the divisor

$$-3K_Z - \Delta \sim (3 - \alpha)\Sigma + (6 + 3n - \beta)l$$

is nef, $\alpha \le 6$. Hence,

$$(7 - \alpha)(n\alpha - 2\beta + 14) - 2 \le 11(7 - \alpha) - 2 \le 53.$$

$\square$

5.11. Let us prove (iii). Additionally to 5.1 we now assume that $f \colon W \to Z$ is a $\mathbb{P}^1$-bundle (in particular, W is smooth). Then $W = \mathbb{P}(\mathcal{E})$ for some rank 2 vector bundle $\mathcal{E}$ on Z. Let L be the tautological divisor.

By the relative Euler exact sequence

$$(5.12) \qquad\qquad -K_W = 2L + f^*(-K_Z - c_1).$$

The Hirsch formula gives us

$$(5.13) \qquad\qquad L^2 \equiv L \cdot f^*c_1 - f^*c_2.$$

Combining we obtain $L^3 = c_1^2 - c_2$ and

$$(5.14) \qquad\qquad -K_W^3 = 6K_Z^2 + 2c_1^2 - 8c_2.$$

By the Riemann-Roch Theorem and the Serre duality

$$(5.15) \qquad h^0(\mathcal{E}) + h^0(\mathcal{E} \otimes \det \mathcal{E}^* \otimes \omega_Z) \ge \frac{1}{2}\left(c_1^2 - 2c_2 - K_Z \cdot c_1\right) + 2.$$

Case $Z = \mathbb{P}^2$. Denote $G := f^*h$, where h is a line on $Z = \mathbb{P}^2$. If c_1 even, we can put $c_1 = -2$. Then $L^2 \cdot G = -2$ and $-K_W^3 = 62 - 8c_2$. Assuming $-K_W^3 \geq 64$, we obtain $c_2 \leq -1$. From (5.15) we have $h^0(\mathcal{E}) \geq 1 - c_2 \geq 2$, i.e., $\dim |L| \geq 1$. Since $L^2 \cdot G < 0$, the linear system $|L|$ has a fixed component. On the other hand, $\mathrm{Pic}(W) = \mathbb{Z} \cdot M \oplus \mathbb{Z} \cdot G$. Therefore, $|L - G| \neq \varnothing$. But then

$$0 \leq (-K_W) \cdot (L - G) \cdot G = (2L + 5G) \cdot (L - G) \cdot G = 2L^2 \cdot G + 3L \cdot G^2 = -1,$$

a contradiction.

Assume that $c_1(\mathcal{E})$ is odd. Then the anti-canonical divisor $-K_W$ is divisible by 2: $-K_W = 2D$, where $D = L + \frac{1}{2}f^*(-K_Z - c_1(\mathcal{E}))$. In this case, a general member $D \in |D|$ is a weak del Pezzo surface with Du Val singularities (see [34]). Then $-K_W^3 = 8K_D^2$. Assuming $-K_W^3 > 64$, we obtain $K_D^2 = 9$, $D \simeq \mathbb{P}^2$ and $D_D = -K_D$. Note that the restriction $H^0(W, \mathcal{O}_W(D)) \to H^0(D, \mathcal{O}_D(D))$ is surjective. Therefore, the divisor D is very ample and defines an embedding $V = V_9 \hookrightarrow \mathbb{P}^{10}$ so that fibers f are mapped to lines in $\mathbb{P}^{10}$. Note that $\phi: W \to V$ is an extremal contraction. If ϕ is of type $(\mathbf{1,0})^{\mathbf{0}}$, then V has only terminal Gorenstein singularities. In this case, by Theorem 1.3 we have $-K_W^3 \leq 64$. Assume that $\phi: W \to V$ is divisorial and let E be an exceptional divisor. Then V is $\mathbb{Q}$-factorial and $\rho(V) = 1$. Since $D \simeq \mathbb{P}^2$, D does not meet fibers of ϕ, i.e., $\phi(D) \cap \phi(E) = \varnothing$. This is possible only if $\phi(E)$ is a point. Therefore, V is a cone over $\phi(D)$. In other words, $V \simeq \mathbb{P}(3, 1, 1, 1)$. We get case (*).

Further, we consider the case when $Z \simeq \mathbb{F}_n$ with $n = 0$ or 2. Let Σ and l be the minimal section and a fiber of $\mathbb{F}_n$ respectively. We may assume that $c_1 = c_1(\mathcal{E}) = a\Sigma + bl$ and $c_2 = c_2(\mathcal{E}) = c$. If both numbers a and b are even, we can put $c_1 = 0$. From (5.12) we obtain

$$-K_W = 2L - f^*K_Z = 2D.$$

where $D \sim L + f^*(\Sigma + (1 + n/2)l)$. As above a general member $D \in |D|$ is isomorphic to $\mathbb{P}^2$. On the other hand, there is no dominant morphisms from $\mathbb{P}^2$ to $\mathbb{F}_n$, a contradiction.

Case $Z \simeq \mathbb{P}^1 \times \mathbb{P}^1$. In this case we can permute a and b.

Case: a is odd, b is even. We may assume that $c_1 = -3\Sigma$. From (5.12), (5.13) and (5.14) we get

$$L^2 \cdot f^*\Sigma = 0, \quad L^2 \cdot f^*l = -3,$$
$$-K_W = 2L + f^*(5\Sigma + 2l), \quad -K_W^3 = 48 - 8c.$$

Hence, $c \leq -2$. By (5.15) we have

$$h^0(\mathcal{E}) + h^0(\mathcal{E} \otimes \mathcal{O}(\Sigma - 2l)) \geq \frac{1}{2}(-2c + 3K_Z \cdot \Sigma) + 2 = -c - 1 > 0.$$

Therefore one of the following holds: $|L| \neq \varnothing$ or $|L + f^*\Sigma - 2f^*l| \neq \varnothing$. On the other hand,

$$0 \leq -K_W \cdot L \cdot f^*l = (2L + f^*(5\Sigma + 2l)) \cdot L \cdot f^*l = -1,$$
$$0 \leq -K_W \cdot (L + f^*\Sigma - 2f^*l) \cdot f^*\Sigma < 0,$$

a contradiction.

Case $c_1 = -\Sigma - l$. As above,

$$L^2 \cdot f^*\Sigma = L^2 \cdot f^*l = -1, \quad -K_W = 2L + 3f^*(\Sigma + l),$$
$$-K_W^3 = (2L + f^*(3\Sigma + 3l))^3 = 52 - 8c.$$

This gives us $c \leq -2$. By (5.15) we have

$$h^0(\mathcal{E}) + h^0(\mathcal{E} \otimes \mathcal{O}(-\Sigma - l)) \geq 1 - c > 0.$$

Therefore, $|L| \neq \varnothing$. Since $L^2 \cdot f^*\Sigma = L^2 \cdot f^*l = -1$, $|L|$ has a fixed component. Write $L = F + |M|$, where F is the fixed part. Then $M = f^*Q$, where $Q \sim \alpha\Sigma + \beta l$, α, $\beta \geq 0$. This gives us

$$0 \leq F \cdot (-K_W) \cdot f^*l = (L - f^*(\alpha\Sigma + \beta l)) \cdot (2L + 3f^*(\Sigma + l)) \cdot f^*l = 1 - 2\alpha$$

Hence, $\alpha = 0$. Similarly we get $\beta = 0$ and $F = 0$, a contradiction.

Case $Z \simeq \mathbb{F}_2$.

Case $c_1 = -\Sigma$. Then

$$L^2 \cdot f^*\Sigma = 2, \quad L^2 \cdot f^*l = -1, \quad -K_W = 2L + f^*(3\Sigma + 4l),$$
$$-K_W^3 = (2L + f^*(3\Sigma + 4l))^3 = 44 - 8c_2.$$

Hence, $c_2 \leq -3$. By (5.15) we have

$$h^0(\mathcal{E}) + h^0(\mathcal{E} \otimes \mathcal{O}(-\Sigma - 4l)) \geq 1 - c_2 \geq 4.$$

Hence, $\dim |L| \geq 2$ (because $\dim |L| > \dim |L - f^*(\Sigma + 4l)|$). Since $L^2 \cdot f^*l < 0$, the linear system $|L|$ has a fixed component. Write $|L| = F + |M|$, where F is the fixed and $|M|$ is the moveable part. Then

$$-K_W \cdot (F + M) \cdot f^*l = -K_W \cdot L \cdot f^*l = 1,$$

Hence, $-K_W \cdot M \cdot f^*l \leq 1$ and $-K_W \cdot F \cdot f^*l \leq 1$. Since $-K_W \cdot f^*\Sigma \cdot f^*l = 2$, the divisor $f^*\Sigma$ can not be a fixed component of $|L|$. Therefore, F has exactly one component which must be horizontal and $M \sim f^*(\alpha\Sigma + \beta l)$ for some α, $\beta \geq 0$. Then

$$-K_W \cdot M \cdot f^*l = -K_W \cdot f^*(\alpha\Sigma + \beta l) \cdot f^*l = 2\alpha.$$

Thus, $\alpha = 0$. But then

$$-K_W \cdot F \cdot f^*\Sigma = -K_W \cdot (L - \beta f^*l) \cdot f^*\Sigma = 2 - 2\beta.$$

The only possibility is $\beta = 1$. In this case, $\dim |L| = \dim |M| = \dim |l| = 1$, a contradiction.

Case $c_1 = -l$. Then

$$L^2 \cdot f^*\Sigma = -1, \quad L^2 \cdot f^*l = 0,$$
$$-K_W = 2L + f^*(2\Sigma + 5l), \quad -K_W^3 = (2L + f^*(3\Sigma + 5l))^3 = 48 - 8c_2.$$

Thus, $c_2 \leq -3$. By (5.15) we have

$$h^0(\mathcal{E}) + h^0(\mathcal{E} \otimes \mathcal{O}(-2\Sigma - 3l)) \geq -c_2 + 1 \geq 4.$$

Hence, $\dim |L| \geq 1$. Since

$$-K_W \cdot L \cdot f^*\Sigma = (2L + f^*(2\Sigma + 5l)) \cdot L \cdot f^*\Sigma = -1 < 0,$$

$f^*\Sigma$ is a fixed component of $|L|$. Further,

$$-K_W \cdot (L - f^*\Sigma) \cdot f^*l = (2L + f^*(2\Sigma + 5l)) \cdot (L - f^*\Sigma) \cdot f^*l = 0.$$

This contradicts the fact that $-K_W$ is nef and big.

Case $c_1 = -\Sigma - l$. Then

$$L^2 \cdot f^*\Sigma = 1, \quad L^2 \cdot f^*l = -1,$$
$$-K_W = 2L + f^*(3\Sigma + 5l), \quad -K_W^3 = (2L + f^*(3\Sigma + 5l))^3 = 48 - 8c_2.$$

Hence, $c_2 \leq -3$. By (5.15) we have

$$h^0(\mathcal{E}) + h^0(\mathcal{E} \otimes \mathcal{O}(-\Sigma - 3l)) \geq 1 - c_2 \geq 4.$$

Thus, $\dim |L| \geq 1$. Since $L^2 \cdot f^*l = -1$, the linear system $|L|$ has a fixed component. Write $|L| = F + |M|$, where F is the fixed and $|M|$ is the moveable part. Then

$$-K_W \cdot (F + M) \cdot f^*l = -K_W \cdot L \cdot f^*l = 1,$$

Hence, $-K_W \cdot M \cdot f^*l \leq 1$ and $-K_W \cdot F \cdot f^*l \leq 1$. Since $-K_W \cdot f^*\Sigma \cdot f^*l = 2$, the divisor $f^*\Sigma$ can not be a fixed component of $|L|$. Therefore, F has exactly one component which must be horizontal and $M \sim f^*(\alpha\Sigma + \beta l)$ for some $\alpha, \beta \geq 0$. Then

$$-K_W \cdot M \cdot f^*l = -K_W \cdot f^*(\alpha\Sigma + \beta l) \cdot f^*l = 2\alpha.$$

Thus, $\alpha = 0$. It follows that

$$-K_W \cdot F \cdot f^*\Sigma = -K_W \cdot (L - \beta f^*l) \cdot f^*\Sigma = 1 - 2\beta < 0,$$

a contradiction.

Now we consider cases when Z contains a (-1)-curve.

Lemma 5.16. *If Z contains a (-1)-curve, then there exists a weak Fano threefold W' satisfying assumptions of 5.1, 5.11 and such that $-K_{W'}^3 \geq -K_W^3 + 6$. Moreover, if $-K_{W'}^3 = -K_W^3 + 6$, then the linear system $|-nK_{W'}|$ on W' can contract only f'-vertical divisors.*

Proof. Let $C \subset Z$ be a (-1)-curve, let $\delta \colon Z \to Z'$ be its contraction, and let $D := f^{-1}(C)$. Then $D \simeq \mathbb{F}_n$ for some n. According to Lemma 5.7, for the minimal section Σ of the surface $D \simeq \mathbb{F}_n$ we have $0 \geq K_W \cdot \Sigma = n - 1$. Hence, $n \leq 1$. If $n = 1$, then again by Lemma 5.7 there exists the following commutative diagram

$$
\begin{array}{ccc}
W \dashrightarrow^{\chi} W^+ \xrightarrow{\sigma} W' \\
\downarrow{\scriptstyle f} \qquad\qquad \downarrow{\scriptstyle f'} \\
Z \xrightarrow{\qquad \delta \qquad} Z'
\end{array}
$$

where χ is a flop, σ is an extremal divisorial contraction which contracts a divisor to a smooth point and f' is a $\mathbb{P}^1$-bundle. Then W^+ is again a weak Fano threefold, satisfying assumptions of 5.1 and 5.11. Since σ contacts a divisor to a point, the same holds for W'. It is clear that $-K_W^3 = -K_{W^+}^3 = -K_{W'}^3 - 8$.

The case $n = 0$ can be treated by a similar way using (iii) from Lemma 5.7. Here $\chi = \mathrm{id}$ and σ contracts a divisor S to a fiber of the morphism f'. Therefore, $-K_W^3 = -K_{W'}^3 - 6$. Moreover, if $\sigma(S)$ meets a $K_{W'}$-trivial curve C', then for its proper transform $C \subset W$ we have

$$
K_W \cdot C = \sigma^* K_{W'} \cdot C + S \cdot C > K_{W'} \cdot C' = 0,
$$

a contradiction. $\qquad\square$

Now we finish the proof of Proposition 5.2 in case, when Z contains a (-1)-curve. By inductive hypothesis we may assume that $-K_{W'}^3 \leq 72$. If $-K_W^3 \geq 66$, then $-K_{W'}^3 = 72$ and W' is such as in (*). This contradicts to the second statement from Lemma 5.16. This completes the proof of Proposition 5.2. $\qquad\square$

6. Construction

6.1. As above, let W be a weak Fano threefold having only terminal factorial singularities. Let $\phi \colon W \to V$ be a morphism defined by the linear system $|-nK_W|$ for $n \gg 0$. Assume that $-K_W^3 \geq 72$. We may assume that

(1) W has no extremal rays of type $(\mathbf{3,1})$ and $(\mathbf{3,2})$ (see Propositions 4.11 and 5.2);
(2) W has no extremal rays of type $(\mathbf{2,0})_{\bullet}^-$ and $(\mathbf{2,1})_{\bullet}^-$ (see Proposition 4.5).

Since W is a weak Fano, there exists at least one K-negative extremal ray R on W. By the above, R is of types $(\mathbf{2,1})_{\circ\mathbf{1}}^-$, $(\mathbf{2,1})_{\circ\mathbf{0}}^-$ or $(\mathbf{2,0})_{\circ}^-$.

6.2. Taking into account Corollary 4.9 we get that for V at least one of the following holds:

(A) V is singular along a line Γ, or
(B) V contains a plane Π.

For further inquiry it is very convenient also distinguish the following case:

(0) the variety V has at least one singular non-cDV point P.

Recall that a three-dimensional singularity is said to be *cDV*, if it is hypersurface and locally up to analytic coordinate change is given by an equation $\phi(x, y, z) + t\psi(x, y, z, t) = 0$, where $\phi(x, y, z) = 0$ is an equation of a (two-dimensional) Du Val singularity [28].

Now out theorem is a consequence of the following.

Proposition 6.3. *Let V be a Fano threefold having only canonical Gorenstein singularities. Assume that V satisfies conditions (0), (A), or (B) of 6.2. Then $-K_V^3 \leq 72$. Moreover, if the equality $-K_V^3 = 72$ holds, then V is isomorphic to $\mathbb{P}(3, 1, 1, 1)$ or $\mathbb{P}(6, 4, 1, 1)$.*

If $-K_V^3 > 54$, then according to [35] (see §2) the anti-canonical linear system $|-K_V|$ defines an embedding $V \hookrightarrow \mathbb{P}^{g+1}$. Moreover, its image $V_{2g-2} \subset \mathbb{P}^{g+1}$ is an intersection of quadrics ([35, Lemma 3]). Put $\mathcal{L} := |-K_V|$. Then $\dim \mathcal{L} = g + 1 > 38$. Consider the following linear subsystem $\mathcal{H} \subset \mathcal{L}$:

$$
\begin{array}{lll}
\text{in case (0):} & \mathcal{H} := \{H \in \mathcal{L} \mid H \ni P\}; \\
\text{in case (A):} & \mathcal{H} := \{H \in \mathcal{L} \mid H \supset \Gamma\}; \\
\text{in case (B):} & \mathcal{H} := \{H \mid H + \Pi \in \mathcal{L}\}.
\end{array}
$$

It is clear that

$$
\dim \mathcal{H} = \begin{cases} \dim \mathcal{L} - 1 & \text{in case (0),} \\ \dim \mathcal{L} - 2 & \text{in case (A),} \\ \dim \mathcal{L} - 3 & \text{in case (B).} \end{cases}
$$

Now let $\mathcal{L}_W$ and $\mathcal{H}_W$ be proper transforms of $\mathcal{L}$ and $\mathcal{H}$ respectively.

In case (0), according to [28] there exists at least one exceptional divisor B_i with center at P and discrepancy $a(B_i) = 0$. Write $\phi^* \mathcal{H}_W = \mathcal{H} + B$, where $B = \sum b_i B_i$ is an (integral) nonzero effective exceptional divisor over P. Thus,

$$K_W + \mathcal{H}_W + B \sim 0.$$

Similarly, in all cases we have

$$
\begin{aligned}
(6.4) \qquad K_W + \mathcal{H}_W + B &= \phi^*(K_V + \mathcal{H}) \sim 0, \quad \text{(cases (0) and (A)),} \\
K_W + \mathcal{H}_W + B &= \phi^*(K_V + \mathcal{H} + \Pi) \sim 0, \quad \text{(case (B)),}
\end{aligned}
$$

where B is an integral effective non-zero divisor.

Lemma 6.5. *The image of the variety V under the map $\Phi_{\mathcal{H}}$ given by the linear system $\mathcal{H}$ is of dimension 3.*

Proof. Note that in case (0), the map $\Phi_{\mathcal{H}}$ is nothing but the projection from P. Assume that $\dim \Phi_{\mathcal{H}}(V) \leq 2$. Then V is a cone over $\Phi_{\mathcal{H}}(V)$ with the vertex at P. But in this case the singularity $P \in X$ can not be canonical (see, e.g., [28, 2.14]).

Consider, for example, case (A) (case (B) is considered in a similar way). As above we note that $\Phi_{\mathcal{H}}$ is the projection from the line Γ. If $\Phi_{\mathcal{H}}(V)$ is a curve C, then the variety V is a cone over C with the vertex at Γ. But

then a general member $L \in \mathcal{L}$ is also a cone with the vertex at $L \cap \Gamma$. This contradicts the fact that L is a K3 surface with Du Val singularities. Let now $\Phi_{\mathcal{H}}(V)$ be a surface. Then a general fiber of the map $\Phi_{\mathcal{H}} \colon V \dashrightarrow \Phi_{\mathcal{H}}(V)$ is one-dimensional. On the other hand, fibers of this map are cut out on V by planes Λ passing through Γ. Every such a plane cut out on V a scheme which is an intersection of quadrics. Therefore, $\Lambda \cap V = \Gamma + \Gamma_{\Lambda}$, where Γ_{Λ} is a line and fibers of $\Phi_{\mathcal{H}}$ are such lines Γ_{Λ}. Thus there is a line passing through a general point of V. In this case, it is easy to get the bound $-K_V^3 \le 46$ (see [35, Lemma 5]). $\qquad\square$

From Theorem 2.4 and Inversion of Adjunction [37, 3.3, 9.5] we immediately obtain the following.

Corollary 6.6. *The pair $(V, |-K_V|)$ has only canonical singularities.*

We can write
$$K_W + \mathcal{L}_W = \phi^*(K_V + \mathcal{L}) \sim 0.$$
Hence the pair $(W, \mathcal{L}_W)$ is canonical.

Remark 6.7. According to [28, Cor. 2.14] all the components of B are birationally ruled surfaces.

Lemma 6.8. *In notation 6.1, we can take a modification ϕ so that*
  (i) *the pair $(W, \mathcal{H}_W)$ is canonical, and*
  (ii) *the linear system $\mathcal{H}_W$ is nef.*

Proof. First we consider case (0). According to [28, Th 2.11] there exists a blowup $\phi_1 \colon V_1 \to V$ such that
  (i) the variety V_1 normal;
  (ii) the proper transform H_1 of a general divisor $H \in \mathcal{H}$ is Cartier and has only Du Val singularities;
  (iii) $K_{V_1} = \phi_1^* K_V$ (in particular, K_{V_1} is Cartier and V_1 has only canonical singularities).

By Inversion of Adjunction [37, 3.3, 9.5] the pair $(V_1, \mathcal{H}_1)$ is purely log terminal. Since the linear system $\mathcal{H}_1$ consists of Cartier divisors, the pair $(V_1, \mathcal{H}_1)$ is canonical.

Consider a terminal $\mathbb{Q}$-factorial modification $g_1 \colon W \to V_1$ of V_1. The composition
$$\phi \colon W \xrightarrow{\ g_1\ } V_1 \xrightarrow{\ \phi_1\ } V.$$
is a terminal $\mathbb{Q}$-factorial modification for V. In this case, $K_W = g_1^* K_{V_1}$ and $\mathcal{H}_W = g_1^* \mathcal{H}_1 - E$, where $E \ge 0$. Thus, $K_W + \mathcal{H}_W = g_1^*(K_{V_1} + \mathcal{H}_1) - E$. Since the pair $(V_1, \mathcal{H}_1)$ is canonical, $E = 0$ and so is $(W, \mathcal{H}_W)$. For another terminal modification $\phi' \colon W' \to V$ the map $W \dashrightarrow W'$ is an isomorphism in codimension 1, so it is crepant. By Lemma 3.1 the pair $(W', \mathcal{H}')$ is also canonical.

Since the linear system $\mathcal{H}_W$ has no fixed components, there exists only a finite number of curves $C_i \subset W$ having negative intersection numbers with

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$\mathcal{H}_W$ and these curves are contained in $\mathrm{Bs}\,\mathcal{H}_W \subset \phi^{-1}(P)$. Apply the $\mathcal{H}_W$-Minimal Model Program to W over V. After a finite number of $\mathcal{H}_W$-flops we get a new terminal modification $\phi' \colon W' \to V$ with nef linear system $\mathcal{H}_{W'}$. Replacing W with W' we may assume that $\mathcal{H}_W$ is nef.

In cases (A) and (B) we show that the linear system $\mathcal{H}_W$ has no base points and fixed components. In particular, the pair $(W, \mathcal{H}_W)$ is terminal.

Consider case (A). Since $\mathcal{H}$ is a linear system of hyperplane sections passing through Γ, for the proof of the lemma it is sufficient to show only that the morphism ϕ is decomposed through the blowup of Γ as a reduced subscheme in V.

Since every crepant blowup can be extended to a terminal $\mathbb{Q}$-factorial modification, our assertion is an immediate consequence of the following.

Claim 6.9. *Let V be a threefold with cDV singularities and let $\Gamma \subset \mathrm{Sing}(V)$ be an one-dimensional irreducible component. Assume that the curve Γ is smooth. Then the blowup $\sigma \colon \tilde{V} \to V$ of the curve Γ as a reduced subscheme is crepant. In particular, $\tilde{V}$ is normal and has only cDV singularities.*

Proof. The problem is local. Hence we may assume that V is an analytic neighborhood of some point $P \in \Gamma$. Thus we may assume that V is a hypersurface singularity, given by the equation

$$\psi_0(x, y, z) + t\psi_1(x, y, z) + t^2\psi_2(x, y, z) + \cdots = 0,$$

where $\psi_0(x, y, z) = 0$ is an equation of a Du Val singularity, and the curve Γ is given by the equations $x = y = z = 0$. Since V is singular along Γ, $\mathrm{mult}_{(0,0,0)}\,\psi_k \geq 2$ and since V has only cDV singularities, $\mathrm{mult}_{(0,0,0)}\,\psi_0 = 2$. Then in the chart $x \neq 0$ the variety $\tilde{V}$ is given by the equation

$$\psi_0(x, yx, zx)x^{-2} + t\psi_1(x, yx, zx)x^{-2} + t^2\psi_2(x, yx, zx)x^{-2} + \cdots = 0.$$

Here $\psi_0(x, yx, zx)x^{-2}$ is not divisible by x and $\psi_0(x, yx, zx)x^{-2} = 0$ is an equation of a smooth or Du Val point. This shows that $\tilde{V}$ has only cDV singularities in our chart and the fiber over the point P is one-dimensional. By symmetry the same is true in other charts. Therefore all the components of the exceptional divisor are surjectively mapped to Γ. It is easy to see by the Adjunction Formula that the morphism σ is crepant. $\square$

Case (B) is considered in a similar way. Lemma 6.8 is proved. $\square$

Now run the $K_W + \mathcal{H}_W$-Minimal Model Program. On each step relation (6.4) is kept, so the log divisor $K + \mathcal{H} \equiv -B$ can not be nef. At the end we get a canonical pair $(X, \mathcal{H}_X)$ and $K_X + \mathcal{H}_X$-negative contraction $f \colon X \to Z$ to a lower-dimensional variety Z. Moreover, X has only $\mathbb{Q}$-factorial terminal singularities (however X is not necessarily Gorenstein). By Lemma 3.1 the pair $(X, \mathcal{L}_X)$ is canonical and $\mathcal{L}_X \subset |-K_X|$. The Fano threefold $V = V_{2g-2} \subset \mathbb{P}^{g+1}$ is the image of X under the birational map defined by the linear system $\mathcal{L}_X$. According to (6.4) we have

$$K_X + \mathcal{H}_X + B_X \sim 0.$$

By Lemma 6.5 the image $\Phi_{\mathcal{H}_X}(X)$ is three-dimensional. In particular, the linear system $\mathcal{H}_X$ is not a pull-back of a linear system on Z, i.e., $\mathcal{H}_X$ is ample over Z.

Everywhere below we assume that $-K_V^3 \geq 72$. Then according to (2.3) we have $\dim |-K_V| \geq 38$ and $\dim \mathcal{H} \geq 35$. Hence

$$(6.10) \qquad \dim |-K_X| \geq \dim \mathcal{L}_X = \dim |-K_V| \geq 38,$$

$$(6.11) \qquad \dim |H| \geq \dim \mathcal{H} \geq 35,$$

where $H \in \mathcal{H}_X$ is a general divisor.

For Z there are only the following possibilities: a) Z is a point, b) Z is a curve, and c) Z is a surface. In case b), a general fiber X_η is a nonsingular del Pezzo surface and divisors $H|_{X_\eta}$ and $-(K_{X_\eta} + H|_{X_\eta})$ are ample. Therefore, $X_\eta \simeq \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. Below, in next sections, we consider cases according to the dimension of Z and the type of the fiber X_η. In each case we obtain contradiction with (6.10)-(6.11) or show that in (6.10) equalities hold. In the last case $|-K_X| = \mathcal{L}_X$ and the birational map $X \dashrightarrow V$ is given by the complete linear system $|-K_X|$. Then it is easy to show that, for V, there are only two possibilities as in Example 1.4.

7. Case: Z is a point

7.1. In this section we consider the case, when Z is a point. Then $\rho(X) = 1$ and X is a Fano threefold with $\mathbb{Q}$-factorial terminal singularities. In this situation $\operatorname{Pic} X \simeq \mathbb{Z}$ (see, e.g., [14, Prop. 2.1.2]). Let G be an ample Cartier divisor that generates $\operatorname{Pic} X$. We can write $-K_X \equiv rG$ and $\mathcal{H}_X \sim aG$ for some $r \in \mathbb{Q}$, $r > 0$ and $a \in \mathbb{N}$. Such an r is called the *Fano index* of X. Since the divisor $-(K_X + \mathcal{H}_X)$ is ample, we have $r > a \geq 1$.

Proposition 7.2. *Notation as in* 7.1. *Then* $\dim |-K_X| \leq 34$.

Proof. If X has only Gorenstein singularities, then by Theorem 1.3 we have $\dim |-K_X| \leq 34$. Therefore X has at least one point of index > 1. Below we use the following theorem.

Theorem 7.3 ([31]). *Let X be a Fano threefold with terminal singularities. Assume that X has at least one point of index > 1 and the Fano index $r > 1$. Then there is one of the following embeddings of X into a weighted projective space (we use numeration of* [31]):

[1] $X = X_6 \subset \mathbb{P}(1, 1, 2, 3, i)$, $i = 2, 3, 4, 5, 6$, $r = 1 + 1/i$;
[2] $X = X_4 \subset \mathbb{P}(1, 1, 1, 2, i)$, $i = 2, 3$;
[3] $X = X_3 \subset \mathbb{P}(1, 1, 1, 1, 2)$;
[5] $X = \mathbb{P}(1, 1, 1, 2)$.

Recall that a weighted projective space $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_n)$ is said to be *normalized* if $\gcd(w_0, \ldots, w_{j-1}, w_{j+1}, \ldots, w_n) = 1$ for $j = 0, \ldots, n$. Under this condition, the canonical divisor is computed by the formula $\mathcal{O}_{\mathbb{P}}(K_{\mathbb{P}}) = \mathcal{O}_{\mathbb{P}}(-\sum d_j)$. All weighted projective spaces $\mathbb{P}$ in Theorem 7.3

are nonsingular in codimension 2. Therefore, in cases [1]-[3], the standard the Adjunction Formula $K_X = (K_{\mathbb{P}} + X)|_X$ holds. From the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-K_{\mathbb{P}} - 2X) \longrightarrow \mathcal{O}_{\mathbb{P}}(-K_{\mathbb{P}} - X) \longrightarrow \mathcal{O}_X(-K_X) \longrightarrow 0$$

and the vanishing $H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-K_{\mathbb{P}} - 2X)) = 0$ it is easy to compute the dimension of $|-K_X|$. Here we use the fact that $H^0(\mathbb{P}, \mathcal{O}(d))$ is naturally isomorphic to the component of degree d of the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$ with graduation given by $\deg x_j = w_j$. Then in [1]-[3] the maximal value $\dim |-K_X| = 30$ is achieved in case [1] for $i = 6$. In case [5] we have $-K_X \sim \mathcal{O}(5)$ and $\dim |-K_X| = 33$. $\square$

8. Case: $X_\eta \simeq \mathbb{P}^2$

Since $H^1(Z, \mathcal{O}_Z) = H^1(X, \mathcal{O}_X) = 0$ (by the Kawamata-Viehweg Vanishing Theorem), $Z \simeq \mathbb{P}^1$. Let $H \in \mathcal{H}_X$ be a general element. The divisor $-(K_{X_\eta} + H|_{X_\eta})$ is ample, so $H|_{X_\eta} \simeq \mathcal{O}_{\mathbb{P}^2}(1)$ or $\mathcal{O}_{\mathbb{P}^2}(2)$.

8.1. First we consider the case $H|_{X_\eta} \simeq \mathcal{O}_{\mathbb{P}^2}(1)$. The following fact is well-known (see, e.g., [10]). For convenience of the reader we give the proof.

Lemma 8.2. *If $H|_{X_\eta} \simeq \mathcal{O}_{\mathbb{P}^2}(1)$, then f is a $\mathbb{P}^2$-bundle.*

Proof. Let $S = g^*P$ be an arbitrary fiber. Then $H^2 \cdot S = H^2 \cdot X_\eta = 1$. Therefore S is reduced and irreducible. Since the morphism f is flat, the function $\chi(X_z, \mathcal{O}_{X_z}(H))$ is locally constant. Thus, $\chi(S, \mathcal{O}_S(H)) = \chi(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1)) = 3$. On the other hand, $h^2(S, \mathcal{O}_S(H)) = h^0(S, \omega_S \otimes \mathcal{O}_S(-H)) = 0$. Hence, $h^0(S, \mathcal{O}(H)) \geq 3$. Recall the definition of Δ-genus of a polarized variety $(Y, \mathcal{M})$ (see [9]):

$$\Delta(Y, \mathcal{M}) = \dim Y + \mathcal{M}^{\dim Y} - h^0(Y, \mathcal{M}).$$

It is known that $\Delta(Y, \mathcal{M}) \geq 0$ and in case $\Delta(Y, \mathcal{M}) = 0$ the variety Y is normal and the sheaf $\mathcal{M}$ is very ample [9]. In our case we have $\Delta(S, \mathcal{O}_S(H)) \leq 0$ and $\mathcal{O}_S(H)^2 = 1$. Hence, $S \simeq \mathbb{P}^2$. $\square$

Proposition 8.3. *Let $X \to Z = \mathbb{P}^1$ be a $\mathbb{P}^2$-bundle. Assume that the pair $(X, |-K_X|)$ is canonical. Then $\dim |-K_X| \leq 38$. Moreover, if $\dim |-K_X| = 38$, then $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1})$ and the anti-canonical image of X is the weighted projective space $\mathbb{P}(6, 4, 1, 1)$.*

8.4. We may assume that $X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$, where

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2) \oplus \mathcal{O}_{\mathbb{P}^1}(d_3), \quad d_1 \geq d_2 \geq d_3 = 0.$$

Denote the class of the fiber of f by F and the tautological divisor by M, i.e., a divisor such that $\mathcal{O}_X(M) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Put $d := \sum d_i$. The relative Euler sequence gives us

$$-K_X \sim 3M + (2 - d)F.$$

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Further, put $\mathcal{E}' := \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2)$. Then

$$(8.5) \quad H^0(X, \mathcal{O}_X(-K_X)) = H^0\big(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(3) \otimes f^*\mathcal{O}_{\mathbb{P}^1}(2-d)\big) \simeq$$
$$H^0(\mathbb{P}^1, S^3\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(2-d)) = H^0(\mathbb{P}^1, S^3\mathcal{E}'(2-d)) \oplus$$
$$H^0(\mathbb{P}^1, S^2\mathcal{E}'(2-d)) \oplus H^0(\mathbb{P}^1, \mathcal{E}'(2-d)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2-d)),$$

where

$$H^0(\mathbb{P}^1, S^m\mathcal{E}'(2-d)) = \bigoplus_{i=0}^{m} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(id_1 + (m-i)d_2 + 2 - d)).$$

On the other hand, the pair $(X, |-K_X|)$ is canonical (because $|-K_X| \supset \mathcal{L}_X$). In particular, the linear system $|-K_X|$ has no fixed components and its general member has at worst isolated singularities.

It is clear that $H^0(X, M - d_1 F) \simeq H^0(\mathbb{P}^1, \mathcal{E}(-d_1)) \neq 0$. Therefore there exists an effective divisor $D \sim M - d_1 F$. Since the divisor M is nef, $-K_X \cdot D \cdot M \geq 0$. Using Hirsch formula $M^2 = dM \cdot F$ we can compute $-K_X \cdot D \cdot M$ and obtain

$$2d_2 + 2 - d_1 \geq 0.$$

If now $d_2 = 0$, then $d_1 = d \leq 2$ and by (8.5) we get $\dim |-K_X| = 29$. Assume that $d_2 > 0$. Consider the decomposition $\mathcal{E} = \mathcal{E}' \oplus \mathcal{O}_{\mathbb{P}^1}$ and let $C \subset X$ be the section, corresponding to the surjection $\mathcal{E} \to \mathcal{O}$. A general section $L \in |-K_X| = |3M + (2-d)F|$ has multiplicity ≤ 1 at the general point of C and the curve C is given by vanishing of all sections in $H^0(\mathbb{P}^1, \mathcal{E}') \subset H^0(\mathbb{P}^1, \mathcal{E})$. Therefore, in the sum (8.5), the term $H^0(\mathbb{P}^1, \mathcal{E}'(2-d))$ does not vanish. This means that $d_1 + 2 - d \geq 0$. Thus, $d_2 \leq 2$ and $d_1 \leq 2d_2 + 2 \leq 6$. Using these inequalities we get the following possibilities for (d_1, d_2):

$$(1,1), (2,1), (2,2), (3,1), (3,2), (4,1), (4,2), (5,2), (6,2).$$

By (8.5) we can immediately compute that $\dim |-K_X| \leq 38$. The equality holds only for $(d_1, d_2) = (6, 2)$. The last statement of Proposition 8.3 follows from the Fano construction (see [12, Ch. 4, Remark 4.2]). This proves Proposition 8.3.

8.6. From now on to the end of this section we assume that $X_\eta \simeq \mathbb{P}^2$ and $H|_{X_\eta} \simeq \mathcal{O}_{\mathbb{P}^2}(2)$.

We prove the following.

Proposition 8.7. *In notation 8.6 there exists a $K_X + \mathcal{L}$-crepant birational map $X \dashrightarrow X_0$ onto a $\mathbb{P}^2$-bundle X_0 over Z.*

This implies Theorem 1.5 in case 8.6. Indeed, by Proposition 8.3 we have $\dim \mathcal{L} = \dim \mathcal{L}_{X_0} \leq \dim |-K_{X_0}| \leq 38$. Moreover, if equalities hold, then images of X and X_0 under birational maps defined by linear systems $\mathcal{L}$ and $|-K_{X_0}|$ coincide.

Lemma 8.8. *Let $S = f^*z_0$, $z_0 \in Z$ be a reduced fiber. Then*
  (i) *S is a normal surface;*

(ii) $S \simeq \mathbb{P}^2$ *or* $\mathbb{W}_4$;

(iii) *the pair (X, S) is purely log terminal.*

Proof. As in the proof of Lemma 8.2 we have $h^0(S, \mathcal{O}(H)) \geq 6$ and $\Delta(S, \mathcal{O}_S(H)) = 0$. Therefore the surface S is normal and there is an embedding $S \hookrightarrow \mathbb{P}^5$ whose image is a surface of degree 4 (because $\mathcal{O}_S(H)^2 = 4$). It is well-known that in this situation for S there are only possibilities in (ii). The statement (iii) follows now by Inversion of Adjunction [37, 3.3, 9.5] (because X is nonsingular in codimension 2). $\qquad\square$

Corollary 8.9. *The bundle $f \colon X \to Z$ has no multiple fibers.* [1]

Proof. Assume that $f^* z_0 = mS$ is a fiber of multiplicity $m > 1$. Consider X as a small neighborhood of S and consider the base change

$$
\begin{array}{ccc}
X & \xleftarrow{\ \varphi\ } & X' \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\
Z & \xleftarrow{\ \psi\ } & Z'
\end{array}
$$

where ψ is a covering locally given by $t \to t^m$ and X' is the normalization of the dominant component of $X \times_Z Z'$. Then φ is a finite cyclic covering of degree m which is étale outside of $\operatorname{Sing} X$. Put $S' := \varphi^{-1}(S)$. Then S' is a Cartier divisor and coincides with the scheme-theoretical fiber: $S' = f'^* z_0'$, where $z_0' = \psi^{-1}(z_0)$. Since $\operatorname{Sing} X$ is a finite set, components S' can intersect each other only in a finite number of points $\varphi^{-1}(\operatorname{Sing} X)$. This contradicts Lemma 8.10 below. Therefore the fiber S' is irreducible and reduced. By Lemma 8.8, $S' \simeq \mathbb{P}^2$ or $\mathbb{W}_4$. However a cyclic group acting on $\mathbb{P}^2$ or $\mathbb{W}_4$ has a curve of fixed points. A contradiction shows that $m = 1$. $\qquad\square$

Lemma 8.10. *Let $P \in Y$ be a three-dimensional terminal singularity and let D', D'' be effective Weil divisors such that $D = D' + D''$ is Cartier. Then $\operatorname{Supp} D' \cap \operatorname{Supp} D'' \geq 1$.*

Proof. Assume that $\operatorname{Supp} D' \cap \operatorname{Supp} D'' = \{P\}$. It is clear that we can replace $P \in Y$ with its canonical covering. Thus we may assume that $P \in Y$ is an isolated hypersurface singularity. Then D is a locally complete intersection and $\operatorname{Supp} D' \cap \operatorname{Supp} D'' = \{P\}$ which is impossible. $\qquad\square$

Corollary 8.11. *If $S \simeq \mathbb{P}^2$, then X is nonsingular along S. If $S \simeq \mathbb{W}_4$, then X has exactly one singular point which is the vertex of the cone $\mathbb{W}_4$ and this point is analytically isomorphic to*

(8.12) $\qquad \{x_1 x_2 + x_3^2 + x_4^n = 0\}/\boldsymbol{\mu}_2(1, 1, 1, 0), \quad n \geq 1.$

Proof. If the variety X is singular at some point $P \in S$, then so is the surface S (because S is a Cartier divisor). Assume that $S \simeq \mathbb{W}_4$. Then the (Gorenstein) index of the vertex $O \in S$ is equal to the index of the point

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[1]There is a gap in the proof. See [math.AG/0604468, Lemma 5.9] for corrections.

$O \in X$. Therefore $O \in X$ is a point of index 2 (in particular, this point is singular). Now we consider $X \ni O$ as a small neighborhood and consider the canonical $\boldsymbol{\mu}_2$-covering $X' \to X$ near O. This induces the covering $S' \to S$, where $S' \simeq \mathbb{C}^2/\boldsymbol{\mu}_2(1,1,1)$, i.e., S' is a singularity of type A_1. Since $S' \subset X'$ is a Cartier divisor, we may assume that $X' \ni O'$ is given by the equation $x_1 x_2 + x_3^2 + x_4 \phi(x_1, x_2, x_3, x_4)$ and $\boldsymbol{\mu}_2$ acts on x_1, x_2, x_3 by $x_i \to -x_i$. The rest follows by the classification of terminal singularities. $\square$

Lemma 8.13. *Let $X \ni O$ be a singular point of the form (8.12), let $\sigma\colon \tilde{X} \to X$ be the weighted blowup with weights $\frac{1}{2}(1,1,1,2)$ and let E be the exceptional divisor. Then*

(i) $E \simeq \mathbb{W}_4$ *for* $n \geq 2$ *and* $E \simeq \mathbb{P}^2$ *for* $n = 1$;

(ii) $a(E) = 1/2$;

(iii) *if* $n = 1$, *then the variety* $\tilde{X}$ *is nonsingular, if* $n \geq 2$, *then* $\tilde{X}$ *has exactly one singular point at the vertex* $O_4 \in \mathbb{W}_4$ *which (up to analytic isomorphism) has the form*

$$(8.14) \qquad \{x_1 x_2 + x_3^2 + x_4^{n-1} = 0\}/\boldsymbol{\mu}_2(1,1,1,0).$$

Proof. The divisor E is given in $\mathbb{P}(1,1,1,2)$ by the equation $x_1 x_2 + x_3^2 = 0$ for $n \geq 2$ and $x_1 x_2 + x_3^2 + x_4^2 = 0$ for $n = 1$. This implies (i). Computations of discrepancies in (ii) is the standard toric technique. We prove (iii). The variety $\tilde{X}$ is covered by four affine charts U_i. The map $X \dashrightarrow U_1 \simeq \mathbb{C}^4$ is given by the following formulas

$$(x_1, x_2, x_3, x_4) \longrightarrow (x_1^{1/2}, x_2 x_1^{1/2}, x_3 x_1^{1/2}, x_4 x_1)$$

In this chart $\tilde{X} = \{x_2 + x_3^2 + x_4^n x_1^{n-1}\}$ is nonsingular. Computations for U_2 and U_3 are completely the same.

The map $X \dashrightarrow U_4 \simeq \mathbb{C}^4/\boldsymbol{\mu}_2(1,1,1,0)$ has the form

$$(x_1, x_2, x_3, x_4) \longrightarrow (x_1 x_4^{1/2}, x_2 x_4^{1/2}, x_3 x_4^{1/2}, x_4)$$

Thus, in this chart, $\tilde{X} = \{x_1 x_2 + x_3^2 + x_4^{n-1}\}/\boldsymbol{\mu}_2(1,1,1,0)$ has exactly one singular point at the origin. The rest is obvious. $\square$

To finish the proof of Proposition 8.7 we consider the weighted blowup $\sigma\colon \tilde{X} \to X$ as in Lemma 8.13. Then

$$K_{\tilde{X}} + \tilde{S} = \sigma^*(K_X + S) - \frac{1}{2}E, \quad \sigma^* S = \tilde{S} + E$$

(because S is Cartier and the pair (X, S) is purely log terminal). Therefore the pair $(\tilde{X}, \tilde{S} + \frac{1}{2}E)$ is also purely log terminal. Hence $\tilde{S}$ is a normal surface. Note that the divisor

$$K_{\tilde{S}} = (K_{\tilde{X}} + \tilde{S})|_{\tilde{S}} = -\frac{1}{2}E|_{\tilde{S}} + \sigma^*(K_X + S)|_{\tilde{S}}$$

is $\sigma|_{\tilde{S}}$-ample. Therefore $\tilde{S}$ is the minimal resolution of the singularity $S \simeq \mathbb{W}_4$, so $\tilde{S} \simeq \mathbb{F}_4$. Since the surface $\tilde{S}$ is nonsingular, so is the variety $\tilde{X}$ along $\tilde{S}$. It is clear that $\tilde{S} \cap E$ is the minimal section Σ on $\tilde{S} \simeq \mathbb{F}_4$. Let l be a fiber

of the projection $\tilde{S} \simeq \mathbb{F}_4 \to \mathbb{P}^1$. Note that the intersection $\tilde{S} \cap E$ is reduced at the general point (because the pair $(\tilde{X}, \tilde{S} + \frac{1}{2}E)$ purely log terminal). Then $\tilde{S} \cdot l = \sigma^* S \cdot l - E \cdot l = -\Sigma \cdot l = -1$. According to the contractibility criterion there exists a birational contraction $\varphi \colon \tilde{X} \to X_{n-1}$ over Z which contracts the surface $\tilde{S}$. The curve $\varphi(\tilde{S})$ is contained in the nonsingular locus of X_{n-1}. The variety X_{n-1} has exactly one singular point and this point has the form (8.14). Continuing the process we get the following series of birational transformations over Z:

$$X = X_n \dashrightarrow X_{n-1} \dashrightarrow \cdots \dashrightarrow X_1 \dashrightarrow X_0$$

On the last step the variety X_0 is nonsingular. Thus, $X_0 \to Z$ is a $\mathbb{P}^2$-bundle. This proves Proposition 8.7.

9. Case: $X_\eta \simeq \mathbb{P}^1 \times \mathbb{P}^1$

9.1. In this section we assume that $f \colon X \to Z$ is an extremal Mori contraction with general fiber $X_\eta \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Let $H \in \mathcal{H}_X$ be a general element.

Lemma 9.2 (cf. [20, 3.5]). *There exists an embedding of X into a $\mathbb{P}^3$-bundle over Z such that every fiber X_η is a reduced irreducible quadric (in particular, X is Gorenstein and every Weil divisor on X is Cartier).*[2]

Sketch of the proof. Let $f^* z_0$ be an arbitrary fiber. Similar to Lemma 8.8 and Corollary 8.9 one can prove that $f^* z_0$ is an irreducible reduced normal surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or a quadratic cone $\mathbb{W}_2$. Therefore the divisor H is relatively very ample over Z and defines the desired embedding. $\square$

9.3. Thus there is an embedding $X \hookrightarrow \mathbb{P}$ over Z, where

$$\mathbb{P} = \mathbb{P}(\mathcal{E}), \qquad \mathcal{E} = \bigoplus_{i=1}^{4} \mathcal{O}_{\mathbb{P}^1}(d_i)$$

and $X_\eta \subset \mathbb{P}_\eta$ is a quadric. Put $d = \sum d_i$. We can chose d_i so that $d_1 \geq d_2 \geq d_3 \geq d_4 = 0$. Let M be the tautological divisor on $\mathbb{P}$ and let F be a fiber of the projection $\pi \colon \mathbb{P} \to Z$.

We prove the following.

Proposition 9.4. *If, in notation 9.3, $\dim |-K_X| \geq 38$, then there exists a $K_X + \mathcal{L}$-crepant birational map of X onto a $\mathbb{P}^2$-bundle over Z.*

As in §8 this is sufficient for the proof of Theorem 1.5 in the case $X_\eta \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Recall well-known facts which are consequences of the Hirsch formula and relative Euler sequence.

Lemma 9.5.   (i) $\operatorname{Pic} \mathbb{P} = \mathbb{Z} \cdot M \oplus \mathbb{Z} \cdot F$;
(ii) $M^4 = d$, $M^3 \cdot F = 1$, $F^2 \equiv 0$;
(iii) $-K_{\mathbb{P}} = 4M + (2 - d)F$.

---

[2]There is a gap in the proof. See [math.AG/0604468, Lemma 5.9] for corrections.

Thus $X \sim 2M + rF$ for some $r \in \mathbb{Z}$. Put $G := M|_X$ and $Q := F|_X$. By the Adjunction Formula

$$-K_X = 2G + (2 - d - r)Q.$$

Corollary 9.6. *In the above notation, we have* $\operatorname{Pic} X \simeq \mathbb{Z} \cdot G \oplus \mathbb{Z} \cdot Q$.

Lemma 9.7. *We may assume that* $d + r \geq 3$.

Proof. If $d + r < 2$, then the divisor $-K_X$ is ample (because $|G|$ and $|Q|$ are base point free linear systems). By the classification of three-dimensional nonsingular Fano threefolds we have $\dim |-K_V| \leq \dim |-K_X| \leq 34$.

Now let $d + r = 2$. Then $-K_X = 2G$. In this situation a general member of the linear system $|G|$ has only Du Val singularities (see, e.g., [34]). By the Adjunction Formula $-K_G$ is nef and big. Since $\dim |-K_X| \geq 38$, as in (2.3) we have $-K_X^3 \geq 72$. Hence, $K_G^2 = -\frac{1}{8}K_X^3 \geq 9$. On the other hand, applying Noether formula to the minimal resolution $\tilde{G} \to G$ we get $K_{\tilde{G}}^2 \leq 9$. Moreover, equality holds only if $G = \tilde{G} \simeq \mathbb{P}^2$ (cf. 1.11). But the last contradicts the fact that $G \simeq \mathbb{P}^2$ has no nontrivial morphisms to a curve. $\square$

We can write $-K_X \sim H + B$, where B is an effective divisor. By Corollary 9.6, $H \sim G + \alpha Q$ and $B \sim G - (d + r + \alpha - 2)Q$ for some $\alpha \in \mathbb{Z}$.

Lemma 9.8. $d + 2r \leq 6$.

Proof. Since the divisor H is nef,

$$0 \leq (-K_X) \cdot B \cdot H = 2(6 - d - 2r).$$

This proves the statement. $\square$

Lemma 9.9. $r < 0$.

Proof. Assume that $r \geq 0$. Then $d \leq 6$. Since $R^i f_*(\mathcal{O}_X(H)) = 0$ for $i > 0$ (see Lemma 4.4),

$$h^0(\mathcal{O}_X(H)) = h^0(\mathcal{O}_{\mathbb{P}^1}(\mathcal{E}(\alpha))).$$

If $\alpha \leq 0$, then $h^0(\mathcal{O}_X(H)) \leq h^0(\mathcal{O}_{\mathbb{P}^1}(\mathcal{E})) = d + 4 \leq 10$. If $\alpha \geq 0$, then we have $\alpha \leq 2 + d_1 - d - r$ (because B effective) and

$$h^0(\mathcal{O}_X(H)) = d + 4 + 4\alpha \leq 12 + 4d_1 - 3d - 4r \leq 24.$$

In both cases we have a contradiction with our assumption (6.11). $\square$

The linear system $|M|$ has no fixed components and base points and defines a birational contraction of $\mathbb{P}$ onto a cone in $\mathbb{P}^{d+3}$. The subvariety $C := \mathbb{P}(\oplus_{d_i=0}\mathcal{O}_{\mathbb{P}^1}(d_i))$ is contracted to the vertex of the cone. This subvariety is swept out by curves Γ such that $M \cdot \Gamma = 0$. Since $r < 0$, C is contained in X. If $\dim C = 2$, then we have a contradiction with $\rho(X/Z) = 1$. Therefore C is a curve.

Proof of Proposition 9.4. Since $d + r > 2$, we have $-K_X \cdot C < 0$. Therefore, $C \subset \mathrm{Bs}\,|-K_X|$. Let $\sigma\colon (\tilde{\mathbb{P}} \supset \tilde{X}) \to (\mathbb{P} \supset X)$ be the blowup of C. Then $-K_{\tilde{X}}$ is an ample over X Cartier divisor and the exceptional divisor $E \subset \tilde{X}$ of the contraction $\sigma\colon \tilde{X} \to X$ is irreducible. Since the variety $\tilde{X}$ is nonsingular in codimension 1, it is normal. It is clear that $a(E, \mathcal{H}) = 0$. Therefore the contraction σ is crepant: $K_{\tilde{X}} + \tilde{\mathcal{H}} = \sigma^*(K_X + \mathcal{H})$. Thus X' has only $\mathbb{Q}$-factorial canonical singularities and $\rho(X'/Z) = 2$. Note that the divisor $\sigma^*G - E$ is nef over Z (i.e., it is nef on the fibers of $f \circ \sigma\colon \tilde{X} \to Z$). A general fiber $\tilde{Q}$ of the morphism $f \circ \sigma$ is isomorphic to a blowup of a point on $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Therefore on $\tilde{Q}$ there are exactly two curves $\tilde{C}_1$, $\tilde{C}_2$ having intersection number 0 with $\sigma^*G - E$. These curves are is proper transforms of generators C_1, C_2 passing through the point $Q \cap C$. Therefore the divisor $\sigma^*G - E$ defines a supporting function for an extremal ray on $\overline{\mathrm{NE}}(\tilde{X}/Z)$ and the linear system $|n(\sigma^*G - E)|$ for $n \gg 0$ gives an extremal birational contraction $\psi\colon \tilde{X} \to X'$ over Z. The morphism ψ contracts the divisor swept out by $\tilde{C}_i$ to a curve. This shows that a general fiber of the morphism $X' \to Z$ is isomorphic to $\mathbb{P}^2$. Further, we can write $\sigma^*G - E = \psi^*G'$ for some ample over Z Cartier divisor G' on X'. Let Q' be an arbitrary fiber of the morphism $X' \to Z$. Then $(G')^2 \cdot Q' = (\sigma^*G - E)^2 \cdot \tilde{G} = 1$. As in §8 one can prove that $Q' \simeq \mathbb{P}^2$. Therefore $X' \to Z$ is a $\mathbb{P}^2$-bundle. Since $C \subset \mathrm{Bs}\,|-K_X|$, the map $X \dashrightarrow X'$ is crepant. $\qquad\square$

10. CASE: $\dim Z = 2$

In this section we assume that Z is surface. The proof of the following Lemma is a particular case the proof of Theorem 11.8 in [10]. Let H be a general member of the linear system $\mathcal{H}_X$.

Lemma 10.1. *The surface Z is nonsingular and the contraction $f\colon X \to Z$ is a $\mathbb{P}^1$-bundle.*

Proof. If C_{gen} is a general fiber, then $C_{\mathrm{gen}} \simeq \mathbb{P}^1$, $-K_X \cdot C_{\mathrm{gen}} = 2$ and $H \cdot C_{\mathrm{gen}} = 1$. Let $C = f^{-1}(P)_{\mathrm{red}}$ be an arbitrary fiber with the reduced structure and let $m := H \cdot C$. For an ample divisor A on Z and $a, b \gg 0$, we consider a general divisor $S \in |aH + bf^*A|$. By Bertini's theorem S is a nonsingular surface meeting C transversally at ma points. Therefore the restriction $f|_S\colon S \to Z$ is a finite morphism of degree $ma \geq a$ near C. On the other hand, the degree of $f|_S$ is exactly a (because S meets a general fiber at a points). Thus the morphism $f|_S$ is étale near C and the point $P \in Z$ is nonsingular. From this we immediately infer that f is flat (see [19, 23.1]) and every fiber $f^{-1}(P)$ is a reduced irreducible curve arithmetic genus 0. Hence, $f^{-1}(P) \simeq \mathbb{P}^1$ (in particular, f is smooth). $\qquad\square$

The surface Z is rational (because, for example, $H^1(Z, \mathcal{O}_Z) = H^1(X, \mathcal{O}_X) = 0$ and Z is dominated by a component of the divisor B which is a birationally ruled surface, see Remark 6.7). Put $\mathcal{E} := f_*\mathcal{O}_X(H)$. Then $\mathcal{E}$ is a rank 2 vector bundle and $X \simeq \mathbb{P}(\mathcal{E})$.

We fix the following notation and conventions:

10.2. Let Z be a nonsingular rational surface and let X be the projectiviza-tion of a rank 2 vector bundle $\mathcal{E}$ over Z such that the pair $(X, \mathcal{L} \subset |-K_X|)$ is canonical. Assume that there is a decomposition $-K_X \sim H + B$, where H and B are effective f-ample divisors (sections of f), the divisor H is nef, and the image of the map $\Phi_{|H|}$ given by the linear system $|H|$ is three-dimensional. Assume also that (X, Θ) is a generating 0-pair for some bound-ary Θ.

Proposition 10.3. *In notation 10.2 we have* $\dim |H| \leq 37$. *Moreover, if* $\dim |H| \geq 36$, *then* $\dim |-K_X| = 38$ *and the anti-canonical image of* X *is the weighted projective space* $\mathbb{P}(3, 1, 1, 1)$.

Assume that the surface Z contains a (-1)-curve. We use notation of Lemma 5.7. By this lemma we can construct a sequence of birational trans-formations

$$
\begin{array}{ccccccc}
X = X_1 & \dashrightarrow & X_2 = X' & \dashrightarrow & \cdots \dashrightarrow & X_N \\
\downarrow{\scriptstyle f=f_1} & & \downarrow{\scriptstyle f_2} & & & \downarrow{\scriptstyle f_N} \\
Z = Z_1 & \longrightarrow & Z_2 = Z' & \longrightarrow & \cdots \longrightarrow & Z_N
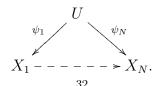\end{array}
$$

where each square is one of transformations (iii)-(v). In case (v) we put $Z_i = Z_{i+1}$. If on some step the invariant $n \geq 2$, then we apply the transformation (v) decreasing n. If $n = 0$ or 1, then we apply the transformation (iii) or (iv) respectively. In these cases the (-1)-curve on the base is contracted and our sequence terminates (i.e., this is possible only on the last step).

Lemma 10.4. *In notation above for the variety* X_N/Z_N *all conditions of* 10.2 *hold. Moreover,* $\dim |H_N| \geq \dim |H|$.

Note however that the canonical property of $(X, |H|)$ is not preserved under out transformations.

Proof. For $n \leq 1$ the map $X' \dashrightarrow X$ does not contract any divisors. By Lemma 3.1 the pair $(X', \mathcal{L}')$ is canonical. For $n \geq 2$, we have $K_X \cdot \Sigma < 0$. Therefore, $\Sigma \subset \mathrm{Bs}\,\mathcal{L}$ and the morphism σ is crepant. By Lemma 3.3 the pair $(X', \mathcal{L}')$ is canonical. So the canonical property of the pair $(X, \mathcal{L})$ is preserved.

Let H_N and B_N be proper transforms of H and B on X_N. Note that the map $\psi \colon X_1 \dashrightarrow X_N$ is an isomorphism on $X_1 \setminus D$ and $\psi(D)$ is contained in a fiber Γ of the projection $f_N \colon X_N \to Z_N$. This shows that $-K_{X_N} \sim H_N + B_N$. Consider a "Hironaka hut"

$$
\begin{array}{ccc}
 & U & \\
{\scriptstyle \psi_1}\swarrow & & \searrow{\scriptstyle \psi_N} \\
X_1 & \dashrightarrow & X_N.
\end{array}
$$

and write
$$H_U = \psi_1^* H - \sum a_i E_i = \psi_N^* H_N - \sum b_i E_i - dD_U,$$
where H_U and D_U are proper transforms of H_1 and D on U, E_i are prime divisors which are exceptional for ψ_1, as well as for ψ_N, and $a_i, b_i, d \geq 0$. Then the divisor
$$\psi_1^* H - \psi_N^* H_N = \sum (a_i - b_i) E_i - dD_U$$
is nef over X_N. By [37, 1.1] we have $a_i \leq b_i$ and $d \geq 0$. Let C be an irreducible curve on X_N and let $\tilde{C} \subset U$ be an irreducible curve dominating C. If $C = \Gamma$, then $H_N \cdot C > 0$ (because $\rho(X_N/Z_N) = 1$). Assume that $C \neq \Gamma$. Since $\psi_N(E_i)$, $\psi_N(D_U) \subset \Gamma$, we have $E_i \cdot \tilde{C} \geq 0$ and $D_U \cdot \tilde{C} \geq 0$. Hence
$$\psi_N^* H_N \cdot \tilde{C} = \psi_1^* H \cdot \tilde{C} + \sum (b_i - a_i) E_i \cdot \tilde{C} + dD_U \cdot \tilde{C} \geq 0.$$
This implies immediately the numerical effectiveness of H_N. The lemma is proved. $\qquad\square$

Corollary 10.5. *In notation 10.2 we may assume that $Z \simeq \mathbb{P}^2$ or $\mathbb{F}_e$, $e \neq 1$.*

Proof. If Z contains a (-1)-curve C, then by Lemmas 5.7 and 10.4 we can contract it. All the properties 10.2 are preserved. $\qquad\square$

Lemma 10.6. *Let $\Gamma \subset Z$ be a nonsingular rational curve such that $\dim |\Gamma| > 0$ and let $\mathcal{E}|_\Gamma \simeq \mathcal{O}_{\mathbb{P}^1}(d_1) + \mathcal{O}_{\mathbb{P}^1}(d_2)$. Then $|d_1 - d_2| \leq 2 + \Gamma^2$.*

Proof. Let $m = |d_1 - d_2|$ and $G := g^{-1}(\Gamma)$. Then $G \simeq \mathbb{F}_m$. We have
$$-2 + m = -2 - \Sigma^2 = K_G \cdot \Sigma = K_X \cdot \Sigma + G \cdot \Sigma = K_X \cdot \Sigma + \Gamma^2.$$
Since the linear system $|-K_X|$ has no fixed components, $K_X \cdot \Sigma \leq 0$ and $m \leq 2 + \Gamma^2$. $\qquad\square$

From now on we assume that $\dim |H| \geq 36$, i.e., $h^0(Z, \mathcal{E}) = \chi(\mathcal{E}, Z) \geq 37$ (see Lemma 4.4). For simplicity we put $c_i := c_i(\mathcal{E})$, $i = 1, 2$. Then
$$-K_X = 2H + f^*(-K_Z - c_1),$$
$$H^2 = H \cdot f^* c_1 - f^* c_2, \qquad H^3 = c_1^2 - c_2.$$
Recall the Riemann-Roch formula for rank 2 vector bundles over a (rational) surface Z:
$$(10.7) \qquad \chi(\mathcal{E}) = \frac{1}{2}(c_1^2 - 2c_2 - K_Z \cdot c_1) + 2.$$
Since $L \cap B$ is an effective 1-cycle, for any nef divisor N on Z we have
$$(10.8) \quad 0 \leq -K_X \cdot B \cdot f^* N =$$
$$(2H + f^*(-K_Z - c_1)) \cdot (H + f^*(-K_Z - c_1)) \cdot f^* N =$$
$$2H^2 \cdot f^* N + 3(-K_Z - c_1) \cdot N =$$
$$2c_1 \cdot N + 3(-K_Z - c_1) \cdot N = -3K_Z \cdot N - c_1 \cdot N.$$

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Hence,

(10.9) $$c_1 \cdot N \le -3K_Z \cdot N.$$

Case: $Z \simeq \mathbb{P}^2$. Taking into account natural isomorphisms $H^{2i}(\mathbb{P}^2, \mathbb{Z}) \simeq \mathbb{Z}$ we assume that c_1 and c_2 are integers. Inequality (10.9) give us $0 \le c_1 \le 9$ and by (10.7)

$$c_1^2 + 3c_1 - 2c_2 \ge 70.$$

Corollary 10.10. *If $\mathcal{E}$ is decomposable, then $\mathcal{E} \simeq \mathcal{O}(3) \oplus \mathcal{O}(6)$ and $\dim | - K_X| = 38$.*

Proof. Let $\mathcal{E} \simeq \mathcal{O}(d) \oplus \mathcal{O}(d + m)$, where $m \ge 0$. Then $d \ge 0$ (because $\mathcal{E}$ is nef). Since $c_1 = 2d + m$ and $c_2 = d^2 + dm \ge 0$,

$$2d^2 + 2dm + m^2 + 6d + 3m \ge 70.$$

Taking into account that $m \le 3$ (Lemma 10.6) and $0 \le 2d + m \le 9$ we get $d = m = 3$. $\qquad\square$

Thus $\Phi_{\mathcal{L}_X}(X) = \mathbb{P}(3, 1, 1, 1)$ if $\mathcal{E}$ decomposable. Further, we assume that $\mathcal{E}$ indecomposable. We claim that $c_1 \le 8$. Indeed, if $c_1 = 9$, then in (10.9) and (10.8) equalities hold. Thus, $-K_X \cdot B \cdot f^*N = 0$ for any ample divisor N on $\mathbb{P}^2$. Then for a general divisor $L \in | - K_X|$, the intersection $L \cap B$ is composed of fibers of f. Therefore the divisor $-K_X$ is nef (otherwise there exists a horizontal curve R such that $L \cdot R < 0$ and $B \cdot R = (L - H) \cdot R < 0$). Let B_0 be a horizontal component of B. It is easy to see that $-K_X \cdot B_0 \cdot f^*N = -K_X \cdot (B - B_0) \cdot f^*N = 0$. Hence $B = B_0$ and B is contracted by the morphism $\Phi_{|-nK_X|}$ to a point. In this case, the vector bundle $\mathcal{E}$ is decomposable. The contradiction shows that $c_1 \le 8$.

10.11. There are two possibilities:

a) c_1 **odd.** Put $c_1 = 2m - 3$. Then $2 \le m \le 5$ and $2m^2 - 3m \ge 35 + c_2$. It is easy to see that

$$c_1(\mathcal{E}(-m)) = -3, \qquad c_2(\mathcal{E}(-m)) = c_2 - m^2 + 3m \le m^2 - 35 < 0.$$

b) c_1 **even.** Put $c_1 = 2m - 2$. Then $1 \le m \le 5$ and $2m^2 - m \ge 36 + c_2$. Here

$$c_1(\mathcal{E}(-m)) = -2, \qquad c_2(\mathcal{E}(-m)) = c_2 - m^2 + 2m \le m^2 + m - 36 < 0.$$

In both cases by the Riemann-Roch formula and Serre Duality we obtain

$$h^0(\mathcal{E}(-m)) + h^0\Big(\mathcal{E}(-m) \otimes \det \mathcal{E}(-m)^* \otimes \mathcal{O}(-3)\Big) \ge \chi(\mathcal{E}(-m)) \ge 1.$$

Therefore, $H^0(\mathcal{E}(-m)) \ne 0$. Let $s \in H^0(\mathcal{E}(-m))$ be a nonzero section. If s does not vanish anywhere, then there is an embedding $\mathcal{O} \hookrightarrow \mathcal{E}(-m)$ and $\mathcal{E}(-m)$ is decomposable. Let $\varnothing \ne Y \subset \mathbb{P}^2$ be the zero locus of s. Since $c_2(\mathcal{E}(-m)) < 0$, $\dim Y = 1$. Choose a general line $\Gamma \subset \mathbb{P}^2$ and let the intersection $\Gamma \cap Y$ consists of k points. Put $r := -c_1(\mathcal{E}(-m))$ ($r = 2$ or 3). Then $\mathcal{E}(-m)|_\Gamma = \mathcal{O}(k) \oplus \mathcal{O}(-r - k)$. By Lemma 10.6 we have $2k + r \le 3$. Hence, $k = 0$ and $Y = \varnothing$, a contradiction.

Case: $Z \simeq \mathbb{F}_e$.

Lemma 10.12. *If $Z \simeq \mathbb{F}_e$, then $e \leq 4$.*

Proof. Indeed, let $f|_L \colon L \to \bar{L} \to Z$ be the Stein factorization. Here $\pi \colon \bar{L} \to Z$ is a double covering, where $\bar{L}$ is a K3 surface having only Du Val singularities. By the Hurwitz formula, $K_{\bar{L}} = \pi^*(K_Z + \frac{1}{2}R)$, where $R \in |-2K_Z|$ is the ramification divisor. But the linear system $|-2K_{\mathbb{F}_e}|$ contains a reduced element only if $e \leq 4$. $\qquad\square$

10.13. Taking into account natural isomorphisms $H^2(\mathbb{F}_e, \mathbb{Z}) \simeq \mathbb{Z} \cdot \Sigma \oplus \mathbb{Z} \cdot l$ and $H^4(\mathbb{F}_e, \mathbb{Z}) \simeq \mathbb{Z}$ we may assume that $c_1 = a\Sigma + bl$ and $c_2 = c$, where a, b, c are integers. From (10.9) and (10.7) we obtain

$$a = c_1 \cdot l \leq -3K_Z \cdot l = 6, \quad b = c_1 \cdot (\Sigma + el) \leq -3K_Z \cdot (\Sigma + el) = 3(2 + e),$$

$$(10.14) \qquad \chi(\mathcal{E}) = -\frac{1}{2}ea(a+1) + ab + a + b - c + 2 \geq 37.$$

Since $\mathcal{E}$ is nef, $c_1 = a\Sigma + bl$ is a nef class, so

$$b \geq ea, \qquad a \geq 0.$$

Put

$$p := \lfloor a/2 \rfloor + 1, \quad q := \lfloor b/2 \rfloor + 1, \quad a' := a - 2p, \quad b' := b - 2q.$$

Then

$$-2 \leq a', b' \leq -1.$$

Consider the twisted bundle $\mathcal{E}' := \mathcal{E} \otimes \mathcal{O}_{\mathbb{F}_e}(-p\Sigma - ql)$. We have

$$(10.15) \qquad c_1(\mathcal{E}') = a'\Sigma + b'l, \quad c_2(\mathcal{E}') = c + eap - aq - bp - ep^2 + 2pq.$$

Claim 10.16.   (i) $c_2(\mathcal{E}') \leq -2$.
    (ii) If $c_2(\mathcal{E}') = -2$, then $a' = -1$.
    (iii) If $c_2(\mathcal{E}') \geq -3$, then $b' = -2$.

Proof. Taking into account (10.14) we get

$$c_2(\mathcal{E}') \leq -\frac{1}{2}ea(a+1) + ab + a + b + 2 - 37 + eap - aq - bp - ep^2 + 2pq =$$

$$-\frac{1}{4}ea^2 - \frac{1}{2}ea + \frac{1}{2}ab + a + b - 35 - \frac{1}{4}ea'^2 + \frac{1}{2}a'b' \leq$$

$$-\frac{1}{4}ea^2 + (4 + e)a - 29 + 3e + \max\{-e + 2, -1/4e + 1\}.$$

If $e = 0$, then

$$c_2(\mathcal{E}') \leq 4a - 29 + 2 \leq -3.$$

Moreover, the equality holds only if $a = 6$. In this case, $b = 6$ and $b' = -2$.
    Let $e = 2$. Then

<div align="center">35</div>

$$c_2(\mathcal{E}') \leq -\frac{1}{2}a^2 + 6a - 23 + \frac{1}{2} < -3.$$

Let $e = 3$. Then

$$c_2(\mathcal{E}') \leq -\frac{3}{4}a^2 + 7a - 20 + \frac{1}{4} < -3.$$

Finally consider the case $e = 4$. Then

$$c_2(\mathcal{E}') \leq -a^2 + 8a - 17 - \frac{1}{4}ea'^2 + \frac{1}{2}a'b' \leq -a^2 + 8a - 17.$$

Assume that $c_2(\mathcal{E}') \geq -3$. Then $3 \leq a \leq 4$ (we keep in mind that $ea \leq b \leq 3(2 + e)$). If $a = 4$, then $a' = -2$ and $c_2(\mathcal{E}') \leq -5 - b'$. Hence $c_2(\mathcal{E}') = -3$ and $b' = -2$. Let $a = 3$. Then $a' = -1$ and $c_2(\mathcal{E}') \leq -3 - b'/2$. Therefore,

$$-3 \leq c_2(\mathcal{E}') \leq -48 + \frac{5}{2}b - \frac{1}{2}b'.$$

This is possible only if $b = 18$, $b' = -2$. $\qquad\square$

Claim 10.17. $\chi(\mathcal{E}') > 0$.

Proof. Put $c' := c(\mathcal{E}')$. By Riemann-Roch we have

$$\chi(\mathcal{E}') = \left(b' - \frac{1}{2}ea'\right)(a' + 1) + a' - c' + 2.$$

Assume that $\chi(\mathcal{E}') \leq 0$. Taking into account Claim 10.16 we obtain

$$\left(b' - \frac{1}{2}ea'\right)(a' + 1) + a' \leq c' - 2 \leq -4.$$

Hence $a' \neq -1$. Therefore, $a' = -2$ and again by 10.16 we have

$$1 - e \leq -b' - e \leq c' \leq -3.$$

Thus, $e = 4$, $b' = -1$ and $c' = -3$. This contradicts Claim 10.16. $\qquad\square$

Claim 10.18. $H^0(\mathcal{E}') \neq 0$.

Proof. Assume that $H^0(\mathcal{E}') = 0$. By Claim 10.17 we have $H^2(\mathcal{E}') \neq 0$. By Serre Duality

$$H^2(\mathcal{E}')^* \simeq H^0(\mathcal{E}'^* \otimes \omega_Z) \simeq H^0(\mathcal{E}' \otimes \det \mathcal{E}'^* \otimes \omega_Z).$$

On the other hand, $(\det \mathcal{E}'^* \otimes \omega_Z)^* = \mathcal{O}_Z((a' + 2)\Sigma + (b' + (e + 2))l)$ and $H^0((\det \mathcal{E}'^* \otimes \omega_Z)^*) \neq 0$, a contradiction. $\qquad\square$

10.19. Consider a nonzero section $s \in H^0(\mathcal{E}')$. If s does not vanish anywhere, then $\mathcal{E}'$ is an extension of some line bundle $\mathcal{E}_1$ by $\mathcal{O}$. But then $c_2(\mathcal{E}') = c_2(\mathcal{E}_1) = 0$. This contradicts Claim 10.16. Therefore, as in the case $Z = \mathbb{P}^2$, the zero locus of s contains a curve Y. Let $Y \sim q_1\Sigma + q_2l$. Then restrictions $\mathcal{E}'$ to general curves $l \in |l|$ and $l' \in |\Sigma + el|$ are

$\mathcal{E}'|_l = \mathcal{O}(q_1) \oplus \mathcal{O}(a' - q_1)$ and $\mathcal{E}'|_{l'} = \mathcal{O}(q_2) \oplus \mathcal{O}(b' - q_2)$. By Lemma 10.6 we have

$$\begin{cases} 2q_1 - b' \leq 2, \\ 2q_2 - a' \leq 2. \end{cases}$$

Since $-a', -b' \geq 1$, this gives us a contradiction. Proposition 10.3 and Theorem 1.5 are proved.

Now we assume that $-K_V^3 = 72$. Then by Proposition 10.3 we may assume that $\dim |H| = 35$. This is possible only in case (B). As above applying Lemmas 5.7 and 10.4, we get the situation when Z contains no (-1)-curves. Therefore, $Z \simeq \mathbb{P}^2$ or $Z \simeq \mathbb{F}_e$, $e = 0, 2, 3, 4$.

First we assume that $Z \simeq \mathbb{P}^2$. According to Proposition 5.2 the divisor $-K_X$ can not be nef. Hence there is a curve R such that $-K_X \cdot R < 0$ and $B \cdot R = -K_X \cdot R - H \cdot R < 0$. Thus $-K_X \cdot B \cdot f^*N > 0$ for any ample divisor N on Z. By (10.8) we get $c_1 \leq 8$. Now the proof is completely similar to 10.11.

Assume now that $Z \simeq \mathbb{F}_e$, $e = 0, 2, 3, 4$. We use notation of 10.13. If $\chi(\mathcal{E}') > 0$, we get a contradiction similar to 10.19. Assume that $\chi(\mathcal{E}') \leq 0$. As above $-K_X \cdot B \cdot f^*N > 0$ for any ample divisor N on Z. Hence $a < 6$ or $b < 3(2 + e)$. Using these inequalities as in Claim 10.16 we obtain the following.

Claim 10.20. *Let $\chi(\mathcal{E}) \geq 36$. Then $c_2(\mathcal{E}') \leq -4$ with the following exceptions:*

   (i) $c_2(\mathcal{E}') \leq -3$, $e = 3$, $a = 5$, $b = 15$, $a' = -1$, $b' = -1$;
   (ii) $c_2(\mathcal{E}') \leq -1$, $e = 4$, $a = 5$, $b = 18$, $a' = -1$, $b' = -2$;
   (iii) $c_2(\mathcal{E}') \leq -1$, $e = 4$, $a = 3$, $b = 18$, $a' = -1$, $b' = -2$;
   (iv) $c_2(\mathcal{E}') \leq -2$, $e = 4$, $a = 4$, $b = 18$, $a' = -2$, $b' = -2$.

By Riemann-Roch

$$0 \geq \chi(\mathcal{E}') = \left(b' - \frac{1}{2}ea' \right)(a' + 1) + a' - c' + 2.$$

It is easy to see now that $a' = -2$ and $0 \geq \chi(\mathcal{E}') = -b' - e - c'$. This is possible only in case (iv) and then $c_2(\mathcal{E}') = -2$, $c_2(\mathcal{E}) = 20$, $\chi(\mathcal{E}') = 0$. Thus, $c_1 = 4\Sigma + 18l$. But then $H^3 = c_1^2 - c_2 = 0$. This contradicts the fact that the divisor H is nef and big. This completes the proof of Theorem 1.5.

References

[1] Alexeev V. *General elephants of $\mathbb{Q}$-Fano 3-folds* // Compositio Math. **91** (1994) 91–116

[2] Borisov A. *Boundedness of Fano threefolds with log-terminal singularities of given index* // J. Math. Sci. Univ. Tokyo, **8** (2001), no. 2, 329–342

[3] Conte A., Murre J. P. *On the definition and on the nature of the singularities of Fano threefolds* // Rend. Sem. Mat. Univ. Politech. Torino **44** (1986) 51–67

[4] Conte A., Murre J. P. *Algebraic varieties of dimension three whose hyperplane sections are Enriques surfaces* // Ann. Sc. Norm. Sup. Pisa **12** (1985) 43–80

[5] Cutkosky S. *Elementary contractions of Gorenstein threefolds* // Math. Ann. **280** (1988) 521-525

[6] Demin, I. V. *Boundedness of the degree of Fano threefolds that are representable by conic bundles* // Uspekhi Mat. Nauk **36** (1981), no. 3, 209–210 (Russian); translation in Russian Math. Surveys **36** (1981), no. 3, 241–242

[7] Fano G. *Sulle varieta a tre dimensioni a curve-sezioni canoniche* // Mem. R. Accad. d'Italia, **8** (1937) 23–64

[8] Fano G. *Su alcune varieta algebriche a tre demensioni aventi curve-sezioni canoniche* // Scritti Mat. offerti a L. Berzolari., Pavia. Tip. Rossetti. (1936) 329–349

[9] Fujita T. *On the structure of polarized varieties with Δ-genera zero* // J. Fac. Sci. Univ Tokyo. Sec. IA. **22** (1975) 103–115

[10] Fujita T. Classification theories of polarized varieties. London Math. Soc. Lect. Notes Series **155**. Cambridge Univ. Press (1990)

[11] Ishii S. *Quasi-Gorenstein Fano 3-folds with isolated nonrational loci* // Compositio Math. **77** (1991), no. 3, 335–341

[12] Iskovskikh V. A. *Anticanonical models of three-dimensional algebraic varieties*, Current problems in mathematics, Vol. **12** (Russian), pp. 59–157, VINITI, Moscow, 1979 (Russian); transl. in J. Soviet Math. **13** (1980) 745–814

[13] Iskovskikh V. A. *On the rationality problem for conic bundles*, Mat. Sb. **182** (1991), no. 1, 114–121 (Russian); transl. in Math. USSR-Sb. **72** (1992), no. 1, 105–111

[14] Iskovskikh V. A., Prokhorov Yu. G. *Fano Varieties*, *Encyclopaedia of Mathematical Sciences* **47**, *Algebraic Geometry V*, eds. A. N. Parshin and I. R. Shafarevich, Springer-Verlag, 1999

[15] Jahnke P., Radloff I. *Generatedness for Gorenstein Fano threefolds with canonical singularities* // preprint (2004)

[16] Kawamata Y. *The crepant blowing-up of 3-dimensional canonical singularities and its application to the degeneration of surfaces* // Ann. Math. **127** (1988) 93–163

[17] Kollár J. *Singularities of pairs* // Proc. Symp. Pure Math. **62** (1995) 221–287

[18] Kollár J., Mori S. *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics **134** (1998)

[19] Matsumura H. *Commutative ring theory*, Cambridge Univ. Press., Cambridge. 1986

[20] Mori S. *Threefolds whose canonical bundles are not numerically effective* // Ann. Math. (2) **115** (1982) 133–176

[21] Mori S. *Flip theorem and the existence of minimal models for 3-folds* // J. Am. Math. Soc. **1** (1988) 117–253

[22] Mori S., Mukai S. *Classifications of Fano 3-folds with $B_2 \geq 2$* // I. Algebraic and topological theories. Papers from the symposium dedicated to the memory of Dr. Takehiko Miyata held in Kinosaki, October 30 – November 9, 1984. Tokyo: Kinokuniya Company Ltd (1986) 496–545

[23] Mukai S. *New developments in the theory of Fano threefolds: Vector bundle method and moduli problems* // Sugaku Expositions **15** (2002) no 2, 125–150

[24] Namikawa Y. *Smoothing Fano 3-folds* // J. Algebraic Geom. **6** (1997) 307–324

[25] Prokhorov Yu. G. *On **Q**-Fano fiber spaces with two-dimensional base* // preprint MPI **95-33** (1995)

[26] Prokhorov Yu. G. *A remark on Fano threefolds with canonical Gorenstein singularities* // to appear in Proc. Fano Conf.

[27] Prokhorov Yu. G., Shokurov V. V. *Toward the second main theorem on complements: from local to global* // unpublished preprint

[28] Reid M. *Canonical 3-folds* // Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, pp. 273–310, Sijthoff & Noordhof

[29] Reid M. *Projective morphisms according to Kawamata* // Univ. of Warwick. Preprint (1983)

[30] Saint-Donat B. *Projective models of K3 surfaces* // Amer. J. Math. **96** (1974) 602–639

[31] Sano T. *Classification of non-Gorenstein $\mathbb{Q}$-Fano d-folds of Fano index greater than $d-2$* // Nagoya Math. J. **142** (1996) 133–143

[32] Sarkisov V. G. *Birational isomorphisms of conic bundles* // Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980) 918–944 (Russian); English transl.: Math. USSR, Izv. **17** (1981) 177–202

[33] Sarkisov V.G. *On conic bundle structures* // Math. USSR, Izv. **20** (1983) 355-390; translation from Izv. Akad. Nauk SSSR, Ser. Mat. **46** (1982) No.2, 371-408

[34] Shin K.-H. *3-dimensional Fano varieties with canonical singularities* // Tokyo J. Math. **12** (1989) 375–385

[35] Cheltcov I. A. *Boundedness of Fano 3-folds of integer index* // Mat. Zametki **66** (1999), no. 3, 445–451 (Russian); transl. in Math. Notes **66** (1999) (2000) no. 3-4, 360–365

[36] Cheltcov I. A. *Three-dimensional algebraic varieties that have a divisor with a numerically trivial canonical class* // Uspekhi Mat. Nauk **51** (1996), no. 1(307), 177–178 (Russian); translation in Russian Math. Surveys **51** (1996), no. 1, 140–141

[37] Shokurov V. V. *3-fold log flips* // Izv. AN SSSR, Ser. mat. **56** (1992), 105–201 (Russian); transl. in Russian Acad. Sci. Izv. Math. **40** (1993), 93–202

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